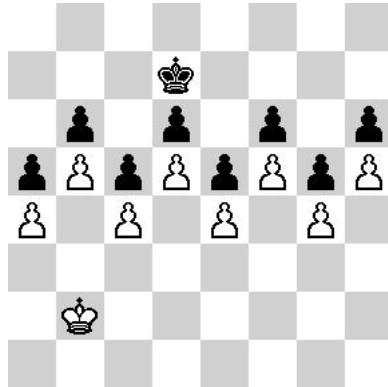


A COMBINATORIAL GAME THEORETIC ANALYSIS OF CHESS ENDGAMES

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1. ABSTRACT

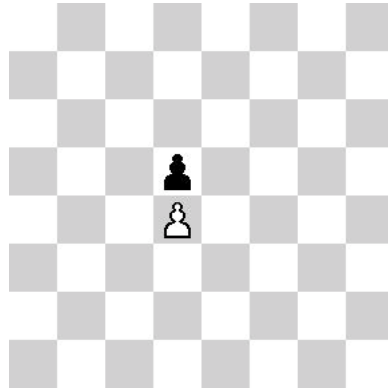
In this paper, we attempt to analyze Chess endgames using combinatorial game theory. This is a challenge, because much of combinatorial game theory applies only to games under normal play, in which players move according to a set of rules that define the game, and the last player to move wins. A game of Chess ends either in a draw (as in the game above) or when one of the players achieves checkmate. As such, the game of chess does not immediately lend itself to this type of analysis. However, we found that when we redefined certain aspects of chess, there were useful applications of the theory. (*Note: We assume the reader has a knowledge of the rules of Chess prior to reading. Also, we will associate Left with white and Right with black*).

We first look at positions of chess involving only pawns and no kings. We treat these as combinatorial games under normal play, but with the modification that creating a passed pawn is also a win; the assumption is that promoting a pawn will ultimately lead to checkmate. Just using pawns, we have found chess positions that are equal to the games 0, 1, 2, \star , \uparrow , \downarrow , and Tiny 1. Next, we bring kings onto the chessboard and construct positions that act as game sums of the numbers and infinitesimals we found. The point is that these carefully constructed positions are games of chess played according to the rules of chess that act like sums of combinatorial games under normal play. Key to the construction of these positions is the placing of the kings in what we call a trebuchet position, in which the first player to move loses the entire game of chess. We will explain this position in

more depth later in the paper, but the basic idea is that a trebuchet position is equal to the zero game, and thus behaves nicely in game sums.

We conclude our paper by making a couple basic rules that one can follow in the endgame to achieve greater success, and finally analyzing an interesting chess puzzle that is easily solved with our theory.

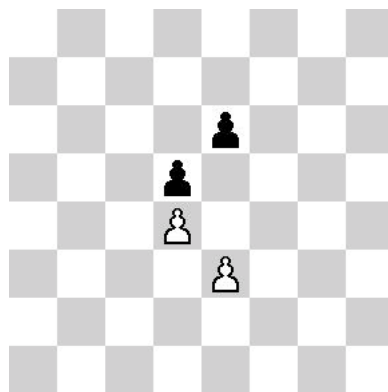
2. THE ZERO GAME



Position 2.1

In Combinatorial Game Theory, the zero game, or 0, is defined as the game where neither player can make any move. Under normal play, the zero game is a second player win, because the first player cannot make any moves. In standard notation, $0 = \{\|\}$. The game above, which features two deadlocked pawns at midboard, is equal to the zero game under normal play, because neither black nor white have any moves.

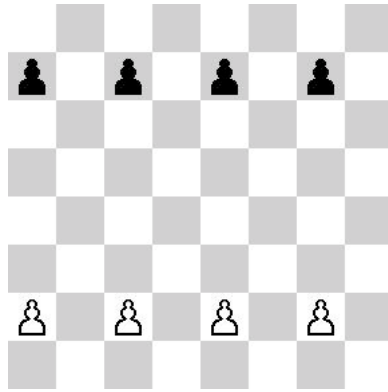
Now consider this game:



Position 2.2

If white is to move, the result is a lost pawn, and black creates a passed pawn which will promote. If Black is to move, the outcome is the opposite. At this point, we feel it is sensible to define creating a passed pawn that will promote as winning the game. With this definition in mind, we see that the position 1.2 is a second player win, and therefore must be equal to the zero game.

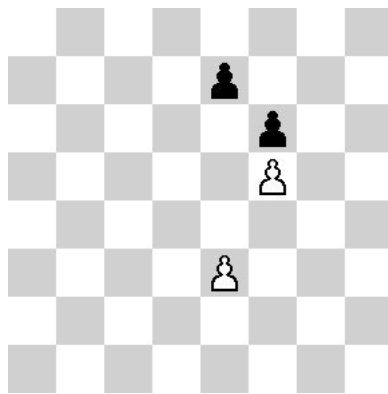
With those positions in mind, it is worth asking whether other chess positions are equal to the zero game. The answer is yes, and many of the zero positions we found exhibit symmetrical characteristics. Take for example, the following pawn position, which is a second player win under normal play:



Position 2.3

It is important to recognize positions that are equal to the zero game because these are second player wins—hence, you should avoid playing in them.

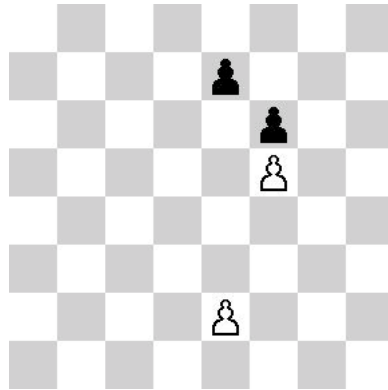
3. SOME NONZERO NUMBERS: ± 1 AND ± 2



Position 3.1: the game 1.

In the game 1, Left can move to 0 and Right has no moves, so 1 is denoted $\{0||\}$. Hence, to construct the game 1 we first looked at a 0 position and modified it as can be seen in position 3.1. White can move to the zero game, but black's only moves lead to a lost e-pawn and a new white queen. In order for this game to completely make sense as 1, we will make another definition: we'll say that having only losing moves is the same as having no moves. This allows us to represent the above position as $\{0||\}$ and call it 1.

Given that the above position is equal to 1, we can now easily create a position equal to 2 by moving white's e-pawn back to the second rank:

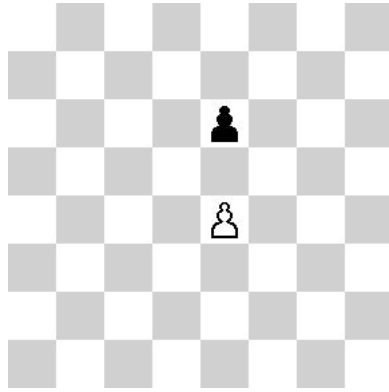


Position 3.2: the game 2.

2 is denoted $\{0, 1||\}$, and you can see that in the above position white can move to both 0 and 1, while black has only losing moves. The reader should note that it is now easy to construct -1 and -2 by switching the white and black pieces and rotating the board 180 degrees.

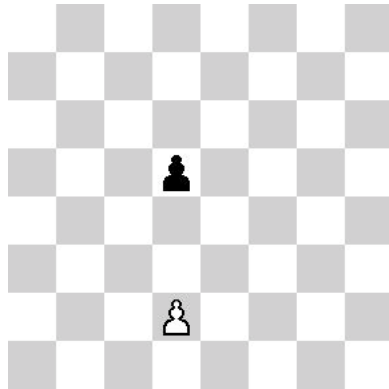
4. SOME INFINITESIMALS: \star , \uparrow , \downarrow

\star is the game in which both left and right can move to the zero game. It is represented formally by $\{0||0\}$. We found the following position to be equal to \star :



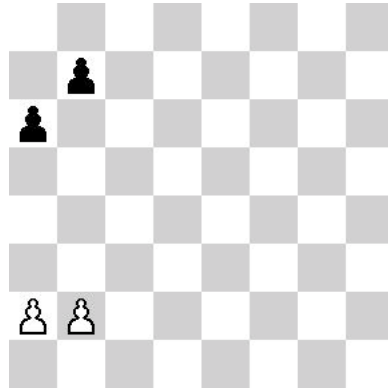
Position 4.1: the game \star .

The game \uparrow is denoted $\{0|\star\}$, and is a positive infinitesimal. Here is a position equal to \uparrow :



Position 4.2: the game \uparrow .

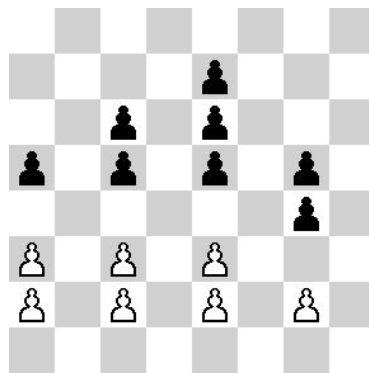
If we denote this game in the standard way, we get $\{0, \star|\star\}$, but \star is a reversible option for white. This is because the original game is greater than zero, and if white plays to \star then black will win by moving to 0. Hence, the game's canonical form is $\{0|\star\}$, which equals \uparrow . \downarrow can be constructed in the same way we described for obtaining -1 from 1.



Position 4.3: this game also equals \uparrow .

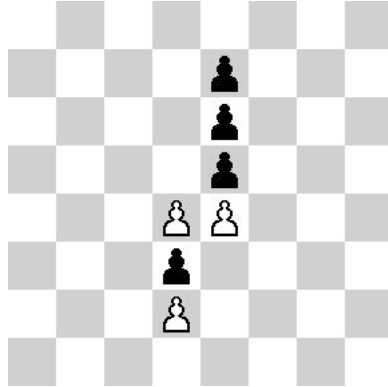
Position 4.3 above is also equal to \uparrow . An evaluation of the game tree will soon give us options that can be pruned by canonicalization. For white, a3 results in a symmetric position, which reduces to zero, because white can mirror any black reply with a pawn move of its own, eventually reducing to a mutual zugzwang. White can also force a zero position with b4, as black is forced to respond with b6, leading to white playing a4 and into a symmetrical game. Whites other move, a4, results in the hot game of $\{2||0\}$ with black to play, which is a reversible option, so it can be pruned from the game tree. 1. b3 a5 2. a3 a4, loses for white, making b3 also a prunable option. Black's options are a little more limited. Playing b5 results in white being able to move to the 1 game, which black certainly does not want, and b6 results in a position that is greater than zero. The only move black really has is a5, which moves the game into a star position. This is because if black could move again to a4, then black moves to the zero game, whereas whites response to a4 would also result in the zero game. Thus, it is shown that this position can indeed be canonicalized to $\{0||\star\}$, or \uparrow .

5. TINY 1



Position 5.1

To construct Tiny 1, we look at games like the four pawn formations in position 5.1. The reader can check that, from left to right, the pawn formations are equal to $\{1||0\}$, $\{1||-1\}$, $\{1||-2\}$, and $\{0||-1\}$. Tiny 1 is denoted $\{0||\{0||-1\}\}$, so in Tiny 1 white will be only be able to move to the zero game, while black's option will be to move to the rightmost pawn position above. Here is the position:

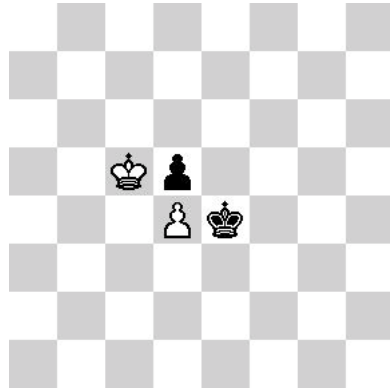


Position 5.2: the game Tiny 1.

White's only reasonable move is to take black's pawn on the fifth rank, leading to a deadlocked zero position (note that white's option to advance the d-pawn is dominated because it leads to a passed pawn for black). Black can only take white's pawn on d4, which leaves the position $\{0||-1\}$, as desired. Tiny 1 is a positive infinitesimal, which means it should be a white win, which this position is. Tiny 1 is also a particularly small infinitesimal—it is infinitesimal with respect to \uparrow . Later, when we discuss game sums, we will show that this position is indeed smaller than \uparrow .

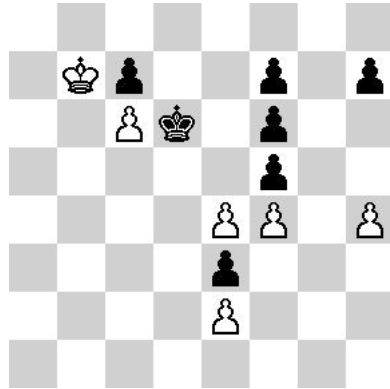
6. THE TREBUCHET POSITION, AND GAME SUMS

Now we will bring kings onto the board. The trebuchet is an example of a reciprocal zugzwang position—a position in which both black and white would prefer not to move. In Position 6.1, the side to move loses the game, so we can say it is equal to the zero game.



Position 6.1: Trebuchet.

It is easily checked that either side to move loses its pawn, and with correct play the other side's pawn will advance and promote, winning the game. Since this position equals the zero game, we can now combine it with the other positions we have previously constructed in order to form positions of chess that are combinatorial games under normal play. For example, let's look at what happens when we combine the trebuchet, Tiny 1 and \downarrow :



Position 6.2: Trebuchet, Tiny 1, and \downarrow .

Tiny 1 is a positive infinitesimal, \downarrow is a negative infinitesimal, and the trebuchet is equal to zero. However, since Tiny 1 is infinitesimal with respect to both \uparrow and \downarrow , we would expect this position to be less than 0, or a win for black. It turns out that this is the case. If white is to move, white can move either in Tiny 1 or down. If white moves in Tiny 1, only the game \downarrow remains. Black then moves \downarrow to 0, and white is forced to play in the trebuchet and loses. If, on the other hand, white first moves \downarrow to \star , then black's winning response is to take white's pawn in Tiny 1, leaving the games $\{0 \parallel -1\}$ and \star . There are two remaining moves before someone has to move in the trebuchet, and white must move, so white loses. If black moves first, the winning move is to take white's pawn in Tiny 1. This shows that our position is indeed less than 0. Of course, knowing our theory makes computing the

outcome class of the position much quicker, because of the fact

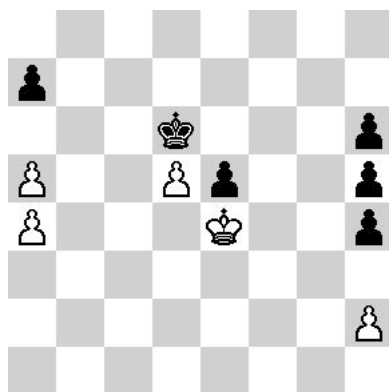
$${}^t1 + \downarrow < 0.$$

7. STRATEGIES IN GAME SUMS

Playing in game sums in chess endgames, we will advise two principles.

Principle 1 The number avoidance theorem says that in a game which is a sum of numbers and games that are not numbers, if a player can win, s/he can do it by not moving in the numbers. Hence, in chess endgames, when faced with a sum of games that are numbers and games that are not, do not move in the numbers.

Principle 2 When given the opportunity, one should move in the game with maximal temperature. The temperature of infinitesimals is 0, and the temperature of a switch game $\{x||y\}$, $x > y$, is $\frac{(x-y)}{2}$. This strategy is known as Hotstrat in Combinatorial Game Theory. We offer the following example as a demonstration:



Position 7.1

In position 7.1, we have two pawn switch games and the kings are involved in a trebuchet, which we'll think of as the zero game. Principle 1 says that neither player should move in 0. Principle 2 says to play in the hottest game, so let's calculate the temperature of the two switch games. The pawn position on the left is equal to $\{1||0\}$, so its temperature is

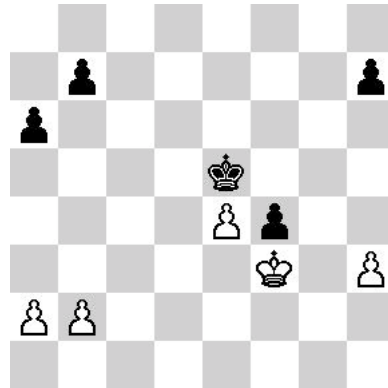
$$\frac{(1 - 0)}{2} = \frac{1}{2}.$$

The pawn position on the right is $\{0|| - 2\}$, which has temperature

$$\frac{(0 + 2)}{2} = 1.$$

Principle 2 says to move in the game on the right. We can see that black will win this game no matter what, but white can lose by less pawn moves by moving to h2 instead of a6.

8. ONE MORE THING: A PUZZLE



Position 8.1: first to move wins.

Here is an application to a real game in which both sides seem evenly matched. Combinatorial game theory can be used to solve position 8.1 (Example 87 in *A Guide to Chess Endings* by Euwe and Hooper). The black and white kings are locked in a trebuchet position and thus do not want to have to move, as moving would lead to the concession of a pawn—critical in an endgame like this one. Hence, we'll treat the trebuchet as a zero game and look at the sum of the two pawn games. The pawn game on the queenside is just position 4.3, which is equal to \uparrow , whereas the game on the kingside canonicalizes to $\downarrow \star$ ($\{\downarrow \parallel 0, \star\}$). Adding the respective values of the game sums gives us $\downarrow \star$, which is confused with zero. This means that whoever is first to play should be able to make a winning pawn move. If white is first to play, white plays 1. h4, which moves to \downarrow in the local pawn game, but moves to zero in the entire game, because the other position is equal to \uparrow . The game sum is now a second player win with Black to play. Black playing 1. ...a5 first achieves a similar effect, resulting in a black win. In terms of chess intuition, one can explain the merits of a5 and h4 as breaking up the opponents move choice, as pawns can choose to either move up one or two spaces on the first move. This fact is invaluable for wasting turns, so white wants to play 1. h4 and then 2. h5 immediately following, just as black wants to play 1. ...a5 and then 2. ...a4 next turn.

Undoubtedly, one can find many more interesting, fairly straightforward applications of combinatorial game theory to the chess endgame. However, this is the end of the road for our paper. We would like to acknowledge that the chessboard diagrams in this paper were generated on Apronus.com. Thanks for reading!