Entering Classes in the College Admissions Model

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Abstract

This note reveals a characteristic of stable matchings in the college admissions problem and provides structural insights and a unified treatment for several results on entering classes in this model, including the famous "Rural Hospital Theorem". We also show that the worst student determines the entire entering class. (JEL C78, D47)

Key words: stable matchings; college admissions; entering classes; lattice; rotation.

1 Introduction

The college admissions problem was introduced by Gale and Shapley (1962), and serves as the cornerstone of many economic studies in school choice (see for example Abdulkadiroğlu and Sönmez 2003).[1] Colleges and students have preferences[2] over each other; and each college can admit students up to its quota while each student can be assigned to at most one college. Stability has been the most commonly used solution concept in this model. It is then important to understand the possible entering classes at stable matchings. Two well-known results of such kind are the following: the Rural Hospital Theorem (Roth 1986) states that any college with unfilled positions in one stable matching is matched to the same set of students in all stable matchings. On the other hand, if a college has multiple entering classes in different stable matchings then it can not be indifferent between different entering classes.[3] And in fact, it must prefer any student in the more preferred entering class to anyone who is in the less preferred entering class, but not in the more preferred one (Roth and Sotomayor 1989). The practical implication of Rural Hospital Theorem is that we can not help unpopular colleges through a mere reselection among stable matchings, while the second result dictates that certain entering classes can not coexist. The goal of this note, is to provide a systematic approach for analyzing entering classes.

When studying the college admissions problem, we often think about another market in which each seat of a college is matched to a single student, instead of each college matching to a set of students. In this market, the preferences are significantly correlated: different

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[1]Sometimes with a slight modification that assumes colleges have priorities instead of preferences, i.e. colleges are not strategic when analyzing incentive issues.

[2]Throughout this paper, we assume colleges have strict responsive preferences (Roth 1985).

[3]Here, only the strict preferences over individual students and responsiveness are assumed. Therefore theoretically it is possible that a college is indifferent between two groups of students.

seats of the same college share the same preference over students; and each student assigns a similar rank for different seats of one college. Such correlations severely restrict the possible changes to entering classes in two consecutive stable matchings:[4] either an entering class does not change, or one student who is worse than all the incumbents is substituted in. This insight not only provides alternative proofs to previously mentioned results in a unified and straightforward manner, but also leads to a few new results: First, the maximum difference of two entering classes of any college provides a lower bound on the number of stable matchings. Second, if two entering classes of one college in two stable matchings have the same least preferred student, then these two entering classes must be exactly the same; in other words, the worst student determines the entire entering class. We could further deduce that the number of possible entering classes for each college is at most linear in the number of students. This is perhaps surprising, as the number of potential entering classes without restrictions, is large. If coupled with results in the large market literature, this bound can be improved to sublinear for a large market. Third, if a student is admitted to a college in one stable matching, and suppose he has a different match in the college optimal stable matching, then there exists a stable matching such that he is the worst student in the entering class of that college. This result guarantees the existence of certain entering classes, instead of placing restrictions like the previous ones. These last two statements indicate that, the worst student appears to play an interesting role in the theory of entering classes.

Since John Conway proved the lattice theorem in the one-to-one marriage model[5] (see Knuth 1976), many efforts have been made in studying stable matchings through lattice theoretic approaches.[6] Blair (1984) verifies the converse proposition: every finite distributive lattice is a set of stable matchings. Therefore the lattice structure itself does not pose much of a restriction on the set of stable matchings. Irving and Leather (1986) analyze the complexity of counting stable matchings, with the help of "rotations". Although the word "lattice" does not appear in their paper, the major tools developed in their analysis, namely the rotations, are closely related to the lattice structure of stable matchings. And this relationship is characterized in Gusfield et al (1987), and Gusfield and Irving (1989), for the standard one-to-one marriage model. In this note, we utilize their framework to analyze the entering classes in a many-to-one college admissions model.

The term "rotation" may not sound familiar to some economics audiences. Informally, it is a rejection chain that comes back to itself. In the man-proposing deferred-acceptance algorithm (Gale and Shapley 1962), a woman can strategically reject some (acceptable) man in the hope that this man will propose to someone else and cause another woman to reject her current partner. Then this newly rejected man will propose again and this process repeats, which forms a chain of rejections. If this rejection chain comes back to itself, i.e. in the end the first woman receives a proposal from a man that she likes better than the one she rejects in the beginning, then she benefits from her manipulation. The concept of rejection chain is widely used in many arguments involving large markets (see for example Immorlica and

[4]More precisely, we are talking about changes among two stable matchings that are adjacent in the Hasse diagram. (Hasse diagrams are defined under Example 2.1, see also `https://en.wikipedia.org/wiki/Hasse_diagram`.)

[5]A special case of the college admissions model when each college has a quota of 1.

[6]A lattice structure presents in a variety of matching models, even when matchings are potentially unstable, see for instance Wu and Roth (2018), Kamada and Kojima (2018).

Mahdian 2005, Kojima and Pathak 2009, and Ashlagi et al 2017). However, sometimes it is hard to track the rejection process precisely: a man can propose to his next most preferred woman and get rejected; then he proposes to his next choice and so on until he is accepted by some woman, or the rejection chain ends. In a rotation, we modify men and women's preferences lists so that a man either only has his current partner on his list, or his next most preferred choice is guaranteed to accept him if he is involved in a rejection chain. We shall see the benefit of this modification in Section 2, and use rotations as building blocks for our analysis in Section 3.

2 Model and Preliminaries

2.1 The College Admissions Model

There is a finite set of colleges $\mathbf{C}$ and a finite set of students $\mathbf{S}$. Each student s has strict preferences $\succ_s$ over the set of colleges and being unmatched, denoted by $\emptyset$. Each college c has a quota q_c and strict preferences $\succ_c$ over individual students.[7] Furthermore its preference over groups of students is **responsive**: for any $S' \subseteq \mathbf{S}$, if $|S'| > q_c$ then $\emptyset \succ_c S'$; if $|S'| \leq q_c$ then for any $s_1 \in \mathbf{S} \backslash S' \cup \{\emptyset\}$, $s_2 \in S'$: $S' \succ_c S' \cup \{s_1\} \setminus \{s_2\}$ if and only if $s_2 \succ_c s_1$. That is, a college c considers a group of students S' as unacceptable if S' contains more students than c's capacity; and any two groups of students that differ in a single student are preference ordered by that of individual students.

A **matching** μ specifies the entering class of each college; more formally, it is a mapping such that: (1) $\forall c \in \mathbf{C}$, $\mu(c) \subseteq \mathbf{S}$, $|\mu(c)| \leq q_c$, and $\forall s \in \mathbf{S}$, $\mu(s) \in \mathbf{C} \cup \{\emptyset\}$; (2) $\mu(s) = c \Leftrightarrow s \in \mu(c)$.[8] We say a matching μ is **individually rational** if: (1) $\forall s \in \mathbf{S}$, $\mu(s) \succsim_s \emptyset$; (2) $\forall c \in \mathbf{C}$, $s \in \mu(c)$, we have $s \succsim_c \emptyset$; and a college-student pair (c, s) **blocks** μ if $c \succ_s \mu(s)$ and at least one of the following situations happen: (1) $\exists s' \in \mu(c)$ such that $s \succ_c s'$; (2) $|\mu(c)| < q_c$ and $s \succ_c \emptyset$. A matching μ is **stable** if and only if it is individually rational and there is no blocking pair. In such a framework, Gale and Shapley (1962) and Roth (1985) show that a stable matching always exists. Furthermore, the set of stable matchings forms a lattice under both the order of common preferences of students, denoted by $\succsim_S$ ($\mu \succsim_S \mu' \Leftrightarrow \mu(s) \succsim_s \mu'(s)$, $\forall s \in \mathbf{S}$), and that of colleges, denoted by $\succsim_C$ ($\mu \succsim_C \mu' \Leftrightarrow \mu(c) \succsim_c \mu'(c)$, $\forall c \in \mathbf{C}$), see Roth and Sotomayor (1990). In fact these two lattices are duals. In other words, colleges and students have opposite interest on stable assignments. In this note we focus on the lattice under the order $\succsim_C$; and the maximum and minimum elements of this lattice are called the **college optimal stable matching** and **student optimal stable matching** respectively (denoted by μ_C and μ_S).

In this model, one commonly used proof technique is to divide each college into multiple copies, each having a quota of one, and make the market one-to-one. More precisely, for each college c_i with quota l, split it into l copies: $c_{i1}, c_{i2}, ..., c_{il}$. Consider a new market ("the related marriage market"), in which the agents are students and copies of colleges. Each

[7] As we mentioned in the beginning, Roth and Sotomayor (1989) show that, assuming strict preferences over individuals and responsiveness, a college can not be indifferent between different entering classes at stable matchings.

[8] We call $\mu(c)$ the entering class of c.

college copy c_{ij} has the same preference over students as c_i; and each student follows his original preference when comparing two copies of different colleges c_i and c_j; and strictly prefers c_{ip} over c_{iq} if and only if $p < q$. Then there is a natural bijection between matchings in the two markets: a matching μ in the many-to-one market, that matches college c_i to $\mu(c_i)$ corresponds to a matching μ' in the related marriage market where c_{ij} is matched to c_i's j-th most preferred student among $\mu(c)$. It is well-known that μ is stable in the many-to-one market if and only if μ' is stable in the related marriage market; moreover, the lattice structures of stable matchings in these two markets are isomorphic (Roth and Sotomayor 1990).[9] In this note we use this isomorphism to study the properties of stable matchings in the college admissions problem.

2.2 Rotations in a One-to-one Matching Market

We now introduce concepts and theorems on rotations in a standard one-to-one matching model. They are not the focus of this note but serve as the building blocks. Therefore we list them without proofs. Readers are referred to Gusfield and Irving (1989) for details. In this subsection, we assume the quota $q_c = 1$ for all colleges $c \in \mathbf{C}$. For convenience (and by convention), we call such a one-to-one matching model a stable marriage problem. As some of the concepts introduced below are rather abstract, we work with the following example for illustration purposes.

Example 2.1. *Consider the following preference lists, taken from Gusfield et al (1987).*

c_1	c_2	c_3	c_4	c_5
s_1	s_2	s_3	s_4	s_5
s_3	s_4	s_1	s_2	s_1
		s_5		

s_1	s_2	s_3	s_4	s_5
c_5	c_4	c_1	c_2	c_3
c_3	c_2	c_3	c_4	c_5
c_1				

These preference lists should be read vertically, for example, c_3's preference is $s_3 \succ_{c_3} s_1 \succ_{c_3} s_5$. Unlisted means unacceptable. One can verify that there are 6 stable matchings: (12345), (14325), (32145), (34125), (32541), (34521). Here, we are using a simplified notation: the number in the i-th position is the index of the student that is matched to c_i; for instance, (32145) means $\mu(c_1) = s_3$, $\mu(c_2) = s_2$, $\mu(c_3) = s_1$, $\mu(c_4) = s_4$ and $\mu(c_5) = s_5$. The set of stable matchings forms a lattice under $\succsim_C$.

In this lattice, a stable matching μ **covers** another stable matching μ', if $\mu \succ_C \mu'$ and there is no stable matching ν such that $\mu \succ_C \nu \succ_C \mu'$. (In other words, μ is the immediate successor of μ'.) A **Hasse diagram** provides a visualization of this lattice (see Figure 1 below): the vertices of this diagram are the stable matchings, and there is a line segment that goes upward from μ' to μ whenever μ covers μ'. The ρ's labeled on the edges are"rotations" to be introduced in Definition 2.5.

Definition 2.2. ***A set of reduced preference lists*** *is a set of preference lists induced from the original ones of a given stable marriage problem with zero or more deletions that satisfy: student s is absent from college c's list if and only if c is absent from s's list. With respect to such a set of lists, we denote $first(x)$, $second(x)$, $last(x)$ to be the first, second, and last person in x's reduced preference list.*

[9] The isomorphism follows from Lemma 5.25 in Roth and Sotomayor (1990).

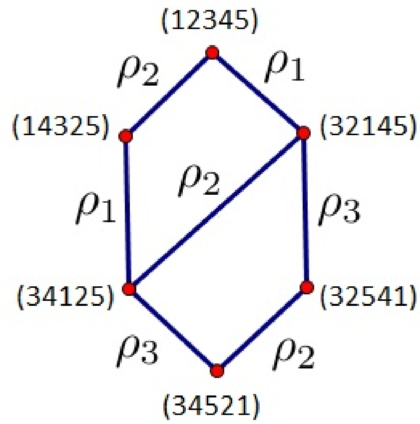

Figure 1: Hasse Diagram for Example 2.1

For example, the original preference lists in Example 2.1 satisfy this definition. We could do some deletions and obtain another reduced preference lists: c_3 only lists s_1 and s_1 only lists c_3, and all other agents have empty lists. In the original lists, $first(c_3) = s_3$, $second(c_3) = s_1$, and $last(c_3) = s_5$.

Definition 2.3. *A set of reduced preference lists is called a **stable set** if for any college c and student s, the following two conditions hold:*
(1) $s = first(c)$ if and only if $c = last(s)$;
(2) s is absent from c's list if and only if $last(s) \succ_s c$.

One can check the original preference lists in Example 2.1 is a stable set.

Proposition 2.4. *There is a bijection between the set of stable matchings and the stable sets.*

Given a stable set, consider matching each college c to $first(c)$. It will be a stable matching. Also, given a stable matching μ, for each student s, remove all the colleges ranked below $\mu(s)$ from s's original preference list, and remove s from c's list if c is removed from s's list, then we get a stable set. This is our desired bijection. And whenever we talk about stable sets and stable matchings interchangeably, we are referring to the bijection specified above.

In Example 2.1, the original preference lists correspond to the stable matching (12345). And from matching (32145), delete c_1 from s_1's list, c_3 from s_3's list, then delete s_1 from c_1's list, s_3 from c_3's list, we obtain a stable set.

Definition 2.5. *A sequence (c_0, s_0), (c_1, s_1), ..., (c_{r-1}, s_{r-1}), $r \geq 2$ is called a **rotation**, denoted by ρ, if with respect to some stable set ψ, $s_{i+1} = first(c_{i+1}) = second(c_i)$ for each i (with $c_r = c_0$ and $s_r = s_0$). And we say ρ is **exposed** in ψ.*

It is not hard to check that in Example 2.1, there are totally 3 rotations: $\rho_1 = (c_1, s_1)$, (c_3, s_3), $\rho_2 = (c_2, s_2)$, (c_4, s_4) and $\rho_3 = (c_3, s_1)$, (c_5, s_5), with ρ_1 exposed in (12345) and (14325); ρ_2 exposed in (12345), (32145) and (32541); ρ_3 exposed in (32145) and (34125).

We can think about rotations in terms of rejection chains. For instance, take the original preference lists in Example 2.1, which correspond to the stable matching (12345). Now start a rejection chain by forcing s_1 to reject c_1, then c_1 proposes to its second choice s_3; s_3 then rejects c_3 in favor of c_1, and finally c_3 proposes to s_1. We have completed the cycle (start from s_1 and back to s_1) and obtained rotation ρ_1.

More generally, suppose $\rho = (c_0, s_0)$, (c_1, s_1), ..., (c_{r-1}, s_{r-1}) is a rotation exposed in ψ. Then the stable matching corresponds to ψ matches c_0 to s_0, c_1 to s_1, ..., and c_{r-1} to s_{r-1}. Start a rejection chain by forcing student s_0 to reject college c_0, then c_0 proposes to its next most preferred student, which is $second(c_0) = s_1$. By the definition of rotation and stable sets, it is always the case that s_1 prefers c_0 to c_1.[10] Then s_1 rejects c_1, and c_1 in turn proposes to its next choice, which is s_2, etc. This rejection process continues, until c_{r-1} is rejected by s_{r-1} and proposes to s_0. Finally s_0 accepts this offer and a rejection cycle has formed.

Conversely, the typical way of finding a rotation is that, start a rejection chain with respect to some stable set; if this rejection chain comes back to itself, then the cyclic part of this rejection chain corresponds to a rotation.

Definition 2.6. *If a rotation ρ is exposed in a stable set ψ, then we define an operation called **rotation elimination** in the following way: for each college c_i, we delete all colleges ranked below c_i from s_{i+1}'s list, and remove s_{i+1} from those corresponding colleges' lists. Then we get a new set of reduced preference lists, call it τ.*

An informal way of describing a rotation elimination is that we simply rematch each college in the rotation to its current second choice and keep other colleges' assignments unchanged (with respect to the stable matching corresponding to ψ). Note that this operation is feasible, as the reassignments only involve students in the rotation. The new matching, which is stable by Proposition 2.7 below, corresponds to the stable set τ. In Example 2.1, if we eliminate $\rho_1 = (c_1, s_1)$, (c_3, s_3) from (12345), we just rematch c_1 to s_3 and c_3 to s_1, which gives us another (stable) matching (32145). In Figure 1, this is characterized by the top right edge between (12345) and (32145), labeled by ρ_1.

Proposition 2.7. *τ is also a stable set.*

In terms of stable matchings, if ρ is exposed in a stable matching μ, then eliminating this rotation leads to another stable matching μ'. Notice the assignment for each college involved in ρ changes from its first choice to second choice, therefore $\mu \succ_C \mu'$.

Proposition 2.8. *For any stable matching μ, it can be obtained by eliminating a sequence of rotations from μ_C, the college optimal stable matching.*

In Example 2.1, (34125) can be obtained from (12345) by eliminating ρ_1 then ρ_2 (or first ρ_2 then ρ_1). This proposition allows us to compare an arbitrary stable matching with μ_C through rotations.

Proposition 2.9.

(I). Suppose two stable matchings μ covers μ', then there is a unique rotation ρ such that μ' can be obtained by eliminating ρ from μ.

[10] We know $s_1 = first(c_1) \Rightarrow c_1 = last(s_1) \Rightarrow c_0 \succ_{s_1} c_1$, as c_0 is on s_1's list, which is implied by $s_1 = second(c_0)$ is on c_0's list.

(II). In the Hasse diagram, if we attach such ρ on the edge between μ and μ' for each such pair, then in any maximal chain between μ_C and μ_S, each rotation shows up exactly once.

(III). For each college c, we can find all of its possible entering classes (or partners, following the tradition of marriage markets) at stable matchings in any maximal chain of the lattice.

Let's demonstrate this proposition with Figure 1. Suppose a stable matching μ, say (14325) covers another stable matching μ', say (34125), then there is an edge between μ and μ' in the Hasse diagram. Statement (I) tells us that μ' can be obtained from μ by eliminating a unique rotation (ρ_1 in this case). Therefore, we can label each edge in the Hasse diagram with a unique rotation.

There are three maximal chains from (12345) to (34521), corresponding to the paths $\rho_2 - \rho_1 - \rho_3$, $\rho_1 - \rho_2 - \rho_3$ and $\rho_1 - \rho_3 - \rho_2$.[11] As statement (II) claims, each rotation shows up exactly once in each of the maximal chains.

In Example 2.1, c_3's potential partners at stable matchings are s_3, s_1, and s_5. Notice $\mu_C(c_3) = s_3$, $\mu_S(c_3) = s_5$, and any maximal chain between μ_C and μ_S must contain either (32145) or (34125), in which c_3 and s_1 are matched.

Statement (I) of Proposition 2.9 characterizes rotations as the difference between stable matchings. It also implies that for any two stable matchings $\mu \succ_C \mu'$, μ' can be obtained by eliminating a sequence of rotations from μ (a generalization of Proposition 2.8). Statement (III) then allows us to study two arbitrary potential partners of any college through rotations, as we can always find them in two matchings along one maximal chain. (See Lemma 3.5 for a formal statement.)

3 Rotations and Entering Classes

Now we are ready to analyze the many-to-one model, using the splitting college trick introduced in Section 2.1. More precisely, given an instance of the college admissions problem, we investigate its rotation structure in the related marriage market, and then use the isomorphism between the lattices of stable matchings (under the common preferences of colleges) in the two markets to prove results in the original college admissions model. Due to the way we construct agents' preferences in the related marriage market, its stable sets also have a special structure:

Theorem 3.1. *Suppose a college c_i has a quota of l, then in any stable set, the reduced preference lists for $c_{i1}, c_{i2}, ..., c_{il}$ have the following structure:*
Suppose the reduced preference list of c_{i1} in this stable set is $s_1, s_2, s_3, ..., s_k$, in a decreasing order. Then the reduced preference list of c_{i2} is exactly $s_2, s_3, ..., s_k$; the reduced preference list of c_{i3} is $s_3, s_4, ..., s_k$; etc. Depending on k and l, the reduced preference lists of c_i look like one of the following triangular arrays:
(the top one is when $k \geq l$, and the bottom one is when $k < l$).

[11] Formally a maximal chain is a chain of matchings, e.g. (12345) – (14325) – (34125) – (34521). By abuse of notation, we also call the path of eliminated rotations along this chain, e.g. $\rho_2 - \rho_1 - \rho_3$, a maximal chain.

c_{i1}	c_{i2}	c_{i3}	...	c_{il}
s_1	s_2	s_3	...	s_l
s_2	s_3	s_4	...	s_{l+1}
s_3	s_4	s_5	...	...
...	...	...	...	s_k
s_{k-2}	s_{k-1}	s_k		
s_{k-1}	s_k			
s_k				

c_{i1}	c_{i2}	c_{i3}	...	c_{ik-1}	c_{ik}	c_{ik+1}	...	c_{il}
s_1	s_2	s_3	...	s_{k-1}	s_k			
s_2	s_3	s_4	...	s_k				
s_3	s_4	s_5	...					
...	...	...	...					
s_{k-2}	s_{k-1}	s_k						
s_{k-1}	s_k							
s_k								

Proof. We show that for any s that is not in c_{i1}'s reduced preference list, it is not in c_{i2}'s reduced preference list either. And for any s other than s_1 that is in c_{i1}'s reduced preference list, it is also in c_{i2}'s reduced preference list. Therefore c_{i2}'s list is exactly $s_2, s_3, ..., s_k$. Repeating this argument, each copy of c_i must have exactly the reduced preference list stated in Theorem 3.1.

If s is not in c_{i1}'s reduced preference list, then by Definition 2.3, $last(s) \succ_s c_{i1}$. By definition we also have $c_{i1} \succ_s c_{i2}$, then by transitivity, $last(s) \succ_s c_{i2}$, therefore s is also absent from c_{i2}'s reduced preference list;

If $s = s_1$, then $last(s_1) = c_{i1} \succ_{s_1} c_{i2}$, therefore it is not in the reduced preference list of c_{i2};

If s is in c_{i1}'s reduced preference list, and $s \neq s_1$, then again by Definition 2.3, $c_{i1} \succ_s last(s)$ (can't be equal since $c_{i1} = last(s_1)$). Notice in s's original preference list, c_{i2} is right below c_{i1}, then $c_{i2} \succsim_s last(s)$, this implies s is in the reduced preference list of c_{i2}. □

The key insights of Theorem 3.1, can be summarized in the following two lemmas (with the same notations as in Theorem 3.1).

Lemma 3.2. *If $k < l$, i.e. if c_i does not fill its quota, then there is no rotation that involves any position (copy) of c_i.*

Proof. Suppose otherwise, say c_{ij} is the first $c_{i1}, c_{i2}, ..., c_{il}$ that is in the rotation. By Definition 2.5, this rotation must contain (c_{ij}, s_j), (c_{ij+1}, s_{j+1}), ..., (c_{ik}, s_k). It can not continue from here to form a cycle, since c_{ik} only has s_k in its reduced preference list, contradiction. □

This provides the intuition for our alternative proof of the Rural Hospital Theorem (Theorem 3.6): If c_i does not fill its quota, then no rotation involves c_i, therefore the entering class of c_i can not change.

Lemma 3.3. *Suppose $k \geq l$, i.e. c_i does fill its quota. Let ρ be any rotation that is exposed in the stable set and involves some position (copy) of c_i. Then eliminating ρ replaces student s_j by s_{l+1} for some j, and all other students matched to c_i are unchanged. (Notice s_{l+1} is less preferred than any incumbent.)*

That is, eliminating a rotation always swaps one incumbent with a student who is worse than all incumbents for each college involved in it.

Proof. Let μ be the stable matching corresponding to the original stable set, and μ' be the stable matching obtained by eliminating ρ from μ. Also, let c_{ij} be the first $c_{i1}, c_{i2}, ..., c_{il}$ that is in ρ, then ρ must contain (c_{ij}, s_j), (c_{ij+1}, s_{j+1}), ..., (c_{il}, s_l). This means, in μ', c_i is matched with $s_1, s_2, ..., s_{j-1}, s_{j+1}, s_{j+2}, ..., s_l, s_{l+1}$; comparing to $\mu(c_i)$, only s_j is replaced by s_{l+1}. □

There are two points in this result: First, the new student is worse than the incumbents, which allows us to study college's preference over students across entering classes (Theorem 3.9 and Theorem 3.10). Second, the size of change is only one, which not only has implications on the difference between entering classes (Theorem 3.7) but also permits results with a flavor of the discrete intermediate value theorem (Theorem 3.13).

The next two lemmas, are direct consequences of the results in the one-to-one marriage model.

Lemma 3.4. *A rotation elimination changes the entering class of c_i if and only if ρ involves some position (copy) of c_i; similarly the college that a student s is admitted to changes if and only if s is involved in ρ.*

Proof. This follows directly from Definition 2.6. □

Lemma 3.5. *For any two entering classes (at stable matchings) $\mu(c_i) \succ_{c_i} \mu'(c_i)$, there exist two stable matchings ν and ν' such that $\mu(c_i) = \nu(c_i)$, $\mu'(c_i) = \nu'(c_i)$, and ν' can be obtained by eliminating a sequence of rotations from ν.*

Proof. By Proposition 2.9 (III), each maximal chain between μ_C and μ_S contains all possible entering classes for any college c_i. Say in one maximal chain, we have matchings ν and ν' such that $\mu(c_i) = \nu(c_i)$ and $\mu'(c_i) = \nu'(c_i)$. Then ν' can be obtained by eliminating a sequence of rotations from ν (along the rotation path of this chain). □

So even though μ and μ' may not be comparable under $\succsim_C$, if we only care about the entering classes of a particular college c_i in μ and μ', we can always find two comparable matchings ν and ν' with the same entering classes as μ and μ' respectively (for c_i). This lemma allows us to compare two arbitrary entering classes through rotations.

The results we prove below follow straightforwardly from the above four lemmas.

Theorem 3.6. (The Rural Hospital Theorem, Roth 1986): *If a college does not fill its quota in a stable matching, then it is matched to the same set of students in any stable matching.*

Proof. Suppose some college c does not fill its quota in some stable matching μ. By Proposition 2.8, it can be obtained by eliminating a sequence of rotations from the college optimal stable matching. Note that, if a college c fills its quota in a stable matching ν, then eliminating any rotation exposed in ν will not create unfilled positions in c; since eliminating a rotation will not make any copy of c change from matched to unmatched. Then we know c does not fill its quota in the college optimal stable matching. Since any other stable matching can be obtained by eliminating a sequence of rotations from the college optimal stable matching, then by Lemma 3.2, all these rotations do not involve c, which means the students matched to c are the same in any stable matchings, by Lemma 3.4.[12] □

Theorem 3.7. (One at a time): *If two stable matchings, μ covers μ', then for any college c, $\mu(c)$ and $\mu'(c)$ differ by at most one student.*

Proof. By Proposition 2.9 (I), μ' can be obtained by eliminating a rotation ρ from μ. Two cases: (1). if ρ does not involve any copy of c, then by Lemma 3.4, $\mu(c) = \mu'(c)$; (2). if ρ involves some copy of c, then by Lemma 3.3, eliminating ρ replaces s_j by s_{l+1}. Therefore no matter what, the difference is at most one student. □

Rotations are sometimes considered as the "minimal" difference between stable matchings, as it identifies the difference between two matchings in a covering relation. Theorem 3.7 reveals that the minimal difference between entering classes (of the same college) is small in a college admissions model.

Remark 3.8. *Theorem 3.7 implies the following: if in two stable matchings μ and μ', the entering classes of a college c, i.e. $\mu(c)$ and $\mu'(c)$, differ by x students, then there are at least $x+1$ stable matchings in this college admissions problem.*

Theorem 3.9. (Extreme favoritism, Roth and Sotomayor 1989): *For two stable matchings μ, μ' and a college c, if $\mu(c) \succ_c \mu'(c)$, then $s \succ_c s'$ for all $s \in \mu(c)$ and $s' \in \mu'(c) - \mu(c)$.*

If a college prefers one entering class to another, then it prefers any student in the better entering class to anyone in the worse entering class, but not in the better one.

Proof. By Lemma 3.5, we can find ν and ν' such that $\mu(c) = \nu(c)$, $\mu'(c) = \nu'(c)$, and ν' can be obtained by eliminating a sequence of rotations from ν. By Lemma 3.4, the students matched to c is unchanged when eliminating a rotation that does not involve c. By Lemma 3.3 if a rotation involves some copies of c, then it will replace student s_j with s_{l+1}, with s_{l+1} being worse than all the incumbents. So all the students added in by rotations are worse than the original students in $\nu(c)$, therefore Theorem 3.9 holds. □

Theorem 3.10. (The mascot): *For two stable matchings μ, μ' and a college c, suppose the least preferred student (under $\succ_c$) in $\mu(c)$ is the same as that of $\mu'(c)$, then $\mu(c) = \mu'(c)$.*

[12] Alternatively, we can prove it through Lemma 3.5. Suppose we want to compare $\mu(c)$ and $\mu'(c)$ with c not filling its quota in one of the matchings. Without loss, we can assume μ' can be obtained from μ through a sequence of rotations (otherwise apply Lemma 3.5). Since eliminating a sequence of rotations does not create new empty positions, it has to be the case that $|\mu(c)| < q_c$, then by Lemma 3.2 all these rotations do not involve c, and $\mu(c) = \mu'(c)$.

So, if an entering class needs a representative, the worst student would be the best candidate.

Proof. It follows from the fact that each rotation elimination involving c will replace the current least preferred student with a worse one (after s_{l+1} joins, s_l is no longer the worst). It also follows from Theorem 3.9: suppose otherwise, without loss, say $\mu(c) \succ_c \mu'(c)$.[13] Let s be the common least preferred student, and $s' \in \mu'(c) - \mu(c)$ (by the Rural Hospital Theorem, Theorem 3.6, it can't be the case that $\mu'(c) \subset \mu(c)$). Then Theorem 3.9 implies $s \succ_c s'$, which contradicts the fact that s is the least preferred student in $\mu'(c)$. □

Corollary 3.11. (Not so many): *The number of entering classes for any college c at stable matchings is at most* $\max\{|\mathbf{S}| - q_c + 1, 1\}$.

Proof. If c does not fill its quota in any stable matching, then by the Rural Hospital Theorem, Theorem 3.6, it has a unique entering class. Otherwise, by Theorem 3.10, each entering class is uniquely determined by the worst student. And the number of students who can potentially be the worst student in an entering class is $|\mathbf{S}| - q_c + 1$. (Without any restriction, the number of possible entering classes is $\binom{|\mathbf{S}|}{q_c}$, which is at the order of $|\mathbf{S}|^{q_c}$.) □

The same logic can be applied to a large market setting. Kojima and Pathak (2009) show that, with short preferences lists for colleges, as the number of colleges n grows to infinity, the expected number of colleges with different entering classes is sublinear in n. Therefore the total number of students with different assignments is also sublinear in n (they assume the quotas of colleges are bounded by a constant). In other words, the total number of students who can potentially become the worst student in any college's entering classes is sublinear in n. Therefore, for a college with multiple assignments, its number of entering classes is sublinear in n. (In comparison, Corollary 3.11 only implies the number of entering classes is at most linear in n.)[14]

Remark 3.12. *One can prove a slightly stronger version of Theorem 3.10 almost verbatim: If the i-th most preferred student in $\mu(c)$ and $\mu'(c)$ is the same, then the first i most preferred students in $\mu(c)$ and $\mu'(c)$ must agree.*[15]

Theorem 3.13. (Don't get cocky): *Let μ be a stable matching. Suppose $s \in \mu(c)$, and $s \notin \mu_C(c)$, then there is a stable matching μ' such that $s \in \mu'(c)$ and s is the least preferred student in $\mu'(c)$.*

So if you are matched to a college, but not among the very best candidates of this college, there is an entering class in which you are the most suitable representative of your cohort.

[13] By footnote 7, c can not be indifferent.

[14] In Kojima and Pathak (2009), with an additional assumption that the market is sufficiently thick, i.e. "the expected number of colleges that are desirable enough, yet have fewer potential applicants than seats, grows fast enough as the market becomes large", they show that all colleges have a unique entering class with high probability. Here, we do not need this assumption, but the result is also weaker.

[15] I do not want to overclaim what is new. As Alvin Roth, one of the authors in Roth and Sotomayor (1989) commented, they were aware of some related results when writing their paper. However, I have not found prior literature emphasizing the importance of the least preferred student, and what's special about the worst student seems likely to be of independent interest. (e.g. see Theorem 3.13.)

Proof. By Proposition 2.8, μ can be obtained by eliminating a sequence of rotations from μ_C. Since $s \notin \mu_C(c)$ and $s \in \mu(c)$, there must exist a rotation ρ involving s such that after its elimination, s is matched to c. Let the first matching after eliminating ρ be μ', then by Lemma 3.3, s must be the least preferred student of c among $\mu'(c)$. □

Remark 3.14. *Again one can prove a slightly stronger result almost verbatim: if s is the i-th least preferred student in $\mu(c)$, then for each $1 \leq j \leq i$, there is a stable matching μ' such that s is the j-th least preferred student in $\mu'(c)$.*

The intuition for Theorem 3.13 and Remark 3.14 is that, as each time when the entering class changes, only one new student is substituted in and becomes the new worst student in the cohort, there has to be a time that s just joined c, and s could only climb up his rank in c one at a time.

4 Conclusion

Rotations are useful in the stable marriage problem: the algorithms for finding the set of all stable matchings are developed from them (Gusfield and Irving 1989). In this note, we show that they also have interesting implications in the college admissions problem. Many results about entering classes, including the Rural Hospital Theorem, can be derived straightforwardly from them. In fact, some new results such as Theorem 3.7 and Theorem 3.13 appear difficult to be proven through standard arguments. And just as a chain is only as strong as the weakest link, the quality of a cohort is determined by its worst student.

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