Forbidden Transactions and Black Markets

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Abstract

Repugnant transactions are sometimes banned, but legal bans sometimes give rise to active black markets that are difficult if not impossible to extinguish. We explore a model in which the probability of extinguishing a black market depends on the *extent* to which its transactions are regarded as repugnant, as measured by the proportion of the population that disapproves of them, and the *intensity* of that repugnance, as measured by willingness to punish. Sufficiently repugnant markets can be extinguished with even mild punishments, while others are insufficiently repugnant for this, and become exponentially more difficult to extinguish the larger they become, and the longer they survive.

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1 Introduction

Why are drug dealers plentiful, but hitmen scarce? I.e. why is it relatively easy for a newcomer to the market to buy illegal drugs, but hard to hire a killer? Both of those transactions come with harsh criminal penalties, vigorously enforced: In the U.S., almost half of Federal prisoners have drug convictions,[1] and murder for hire is treated as murder for both the buyer and the hitman, i.e. both principal and agent.[2] [3]

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[1]See https://www.bop.gov/about/statistics/statistics_inmate_offenses.jsp. Also, in 2018 (the most recent year available), "The highest number of arrests were for drug abuse violations (estimated at 1,654,282 arrests") see https://ucr.fbi.gov/crime-in-the-u.s/2018/crime-in-the-u.s.-2018/topic-pages/persons-arrested.

[2]In the U.S., although murder is generally a State offense, the commercial aspect of murder for hire often qualifies it as a Federal crime under 18 U.S.C. 1958 - USE OF INTERSTATE COMMERCE FACILITIES IN THE COMMISSION OF MURDER-FOR-HIRE, https://www.gpo.gov/fdsys/granule/USCODE-2011-title18/USCODE-2011-title18-partI-chap95-sec1958. Regarding the buyer and the hitman, see U.S. Attorneys' Manual, "1107. Murder-for-Hire—The Offense," https://www.justice.gov/usam/criminal-resource-manual-1107-murder-hire-offense. For 2018, the FBI estimates that there were 14,123 homicides in the U.S. https://ucr.fbi.gov/crime-in-the-u.s/2018/crime-in-the-u.s.-2018/tables/expanded-homicide-data-table-1.xls.

[3]We use murder for hire only as an illustrative example of a market in which it is hard to transact, partly because of the difficulty of trying to gather reliable empirical data on an illegal market that may have few transactions. Note that there are sites on the 'dark web' that claim to offer murder for hire, but seem likely to function as a way to separate the gullible from their bitcoins, see e.g. https://allthingsvice.com/2016/05/14/the-curious-case-of-besa-mafia/.

More generally, many transactions are *repugnant*, in the specific sense that they meet two criteria: some people would like to engage in them, and others think that they should not be allowed to do so (Roth [42]). But only some repugnances become enacted into laws that criminalize those transactions, and only some of those banned markets give rise to active, illegal black markets. Only some of those black markets are so active, yet so difficult to suppress, that the laws banning them are eventually changed so as to allow the transactions that cannot be suppressed to be regulated. Laws that exact harsh punishments but are ineffective at curbing the transactions that they punish may come to be seen as causing harm themselves. Some well-known examples include Prohibition era laws against selling alcohol in the U.S., or laws in much of the world that once banned homosexual sex (and in some places still do).

Markets for opioids (and other prohibited drugs) offer a salient current example. Black markets for drugs are so active and so harmful that many countries have begun to consider whether and how to modify laws that ban them absolutely, or to at least modify the way these illegal marketplaces operate by giving drug users access to legal "harm reduction" resources (such as clean needle exchanges to avoid combining addiction with infection, or safe injection facilities to avoid fatal overdoses[4]). However these proposals for harm reduction also meet with considerable opposition: they are repugnant themselves to many of those who support an absolute ban, and who think that vigorous law enforcement will *eventually* have the desired effect of substantially eliminating the black market (see e.g. Rosenstein [41], by the then deputy attorney general of the United States).

This paper proposes a simple, stylized theoretical model to help understand why some transactions can be relatively quickly eliminated by legally banning them, while others are more resistant, to the point that they may be impossible to extinguish or even suppress to low levels, no matter how long the effort is sustained.

The model will focus on the risks facing a potential entrant to the marketplace: e.g. how risky is it to find a drug dealer, or a hitman? How likely are you to find yourself dealing with a police officer instead? (For contract killing, it appears that there is considerable risk to those seeking a hitman of being arrested before a murder is carried out.[5]) Holding constant the penalties that arise from trying to complete the illegal transaction with an undercover policeman, and the likelihood

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There is also a satirical site that appears to offer hitmen "for rent," and reports having received some inquiries that looked serious enough to report to the authorities, see https://rentahitman.com/. Note further that there are criminal organizations that are capable of murder, which employ hitmen for the purpose (i.e. this is an in-house capability of the organization, rather than one that they purchase on the market; see e.g. Shaw and Skywalker [46], and Brolan, Wilson, and Yardley [11]), and there have been non-employee hitmen: e.g. Schlesinger [44] describes a particular prolific killer who was a contractor to several criminal organizations. Mouzos and Venditto [33] study contract killings in Australia and report that "The category of "contract killing" [that become known to the police] makes up a small percentage of total homicides in Australia (about 2% over a thirteen year period (1989/90 –2001/02)." Reports are rare in the U.S. as well, and successful murders for hire are rare and also seem mostly to involve criminal associates (see e.g. Telford [48] for one such report). Murder itself is relatively rare, see the statistics in the previous footnote.

[4]See e.g. https://harmreduction.org/.

[5]Mouzos and Venditto [33], writing about their subsample of attempted but not completed contract killings, say (p54) "of the 77 incidents examined in this study, 38 were detected through a witness coming forward and then progressed by means of a covert police operation and 37 were detected through a witness coming forward and notifying police of the contract (two incidents did not specify the method of detection)." In contrast, they report that contract killings associated with organized crime are much less likely to be solved (p64): "More than a third of unsolved contract murders were committed as a result of conflict within criminal networks/organised crime (35%), compared with only six percent of solved contract murders." So murders within organized crime seem to be carried out by professionals, but these are apparently much less accessible to people whose motivation for murder is e.g. "Dissolution of a relationship" or "Other domestic," since Mouzos and Venditto report that all of those cases of murder for hire known to the police have been solved. (Once again, we have no way of estimating what part of the market may be missing from these data, e.g. because of hits so professional that they are not noticed to be homicides. . .)

that a random non-criminal citizen will report you to the police if you try to transact with them, the larger the illegal market is, the greater will be the chance of successfully transacting by finding a willing counterparty, and the safer it will be to try to enter it.

We focus on the long run because much of the discussion about whether to modify existing laws and practices focuses on the question of whether continued, consistently vigorous law enforcement will eventually have the desired effect of substantially eliminating the black market, even if the efforts to date have not yet done so. The model has two main results.

First, there are easy and hard cases from the point of view of driving a market to extinction by criminalizing it. The easy cases are those in which the magnitude of the punishment together with the willingness of the population to support the law by reporting and punishing infractions eventually make it too risky for potential new entrants to enter, so that they become law abiding for fear of punishment. The harder case is when the magnitude of the feasible punishment combined with the (un)willingness of the population to support enforcement of the law mean that, if the illegal market is sufficiently large, some portion of the population will be willing to risk entering it. In this case, the eventual extinction of the market will depend on its size, and the probability that the market will remain active enough to sustain itself and can never be extinguished is positive. The second main result says that these hard cases become exponentially harder to extinguish the larger the foothold that the banned market has achieved. This suggests that there may be ways to formally consider the decision problem of when to abandon maximal efforts to extinguish a black market.

Together, these results can help us understand how, when we outlaw some repugnant transactions, we sometimes inadvertently help design self-sustaining black markets. This can inform the discussion of when social policy towards particular repugnant markets should take the form of a "war on drugs," and for which black markets we should consider harm reduction.

Our work is closely related to the literature studying the interaction between law enforcement and social norms. Acemoglu and Jackson [2] develop a dynamic model in which law-breaking is detected in part by whistle-blowing, and discover that "laws that are in strong conflict with existing norms backfire: abrupt tightening of laws causes significant lawlessness, whereas gradual imposition of laws that are more in accord with prevailing norms can successfully change behavior and thus future norms." The main difference between their and our models is that we focus on the population evolution of black markets, instead of scrutinizing individual's law-abidingness. Akerlof and Yellen [3] investigate the relationship between gang crime, law enforcement, and community values, and come to the conclusion that "the traditional tools for crime control-more police cars cruising the neighborhood and longer jail sentences-wrongly applied, will be counterproductive because they undermine community norms for cooperation with the police." Learning from the privatization process in the East European countries and Russia in the 1990s, Hay and Shleifer [29] also argue that "whenever possible, laws must agree with prevailing practice or custom."[6] In a broader sense, our work is also related to the literature concerning the optimal level (and effectiveness) of law enforcement. For instance, see Becker [7], Becker and Stigler [9], Becker, Murphy, and Grossman [8], and a survey by Shavell [45].

There is also an empirical literature on particular black markets that have proved difficult to extinguish. (In this connection, see the exemplary work by Cunningham and Kendall [14][15] and Cunningham and Shah [16][17] on modern markets for prostitution.[7]) The theoretical model we

[6]See also Calvó-Armengol and Zenou [12], and Ferrer [23] for models in which crimes have neighborhood externalities.

[7]On the market for hitmen, see Cameron [13] who focuses on low prices from a very small sample of "amateur" hits, and the citations already mentioned in footnotes; on drugs see Keefer and Loayza [31]; on human organs see Scheper-Hughes [43]. Much of the economic literature on black markets seems to be on prices in markets that evade

explore here is meant to complement empirical work on particular markets as an input to designing possible interventions in those markets.

In terms of technical aspects, our discrete Markov process is connected to the Pólya's urn model - first proposed in Pólya [38], Eggenberger and Pólya [22], and later generalized by Friedman [25]. It has various applications in applied mathematics, see for example the survey by Pemantle [36] and the references there. A unique feature of our model is that we have a decision making process at each time step, therefore we have to understand not only the steady state distribution, but also the realizations along the way. To do this, we combine tools from Markov jump processes, random walks, exponential martingales and optional sampling theory. Moreover, our generalized model discussed in Appendix D is closely related to the stochastic approximation algorithm in Robbins and Monro [40] and processes studied in Hill, Lane, and Sudderth [30] and Pemantle [35].

The rest of the paper is organized as following: Section 2 lays out the model, section 3 explores the conditions under which the black market can be eventually extinguished, and section 4 establishes some results about the likely speed of extinction, and how the probability of extinction decreases quickly as the black market becomes established. Section 5 discusses the kinds of insights we might hope to derive from such a simple model when our attention turns to particular black markets, such as those for prostitution, narcotics, and hitmen, and section 6 concludes. While the model is simple to describe, analysis of the Markov chains it generates requires some care (but Theorem 4.1, which allows us to say that long lived markets are likely to remain so makes it worthwhile). Much of that analysis is presented in the Appendices.

2 The Model

For simplicity of exposition, we will present the model as if the illegal transaction in question involves drugs (but keep in mind other illegal markets, which are met with different degrees of repugnance, like those for murder, prostitution, or horse meat...).

There are 3 types of people. Those belonging to the first type are currently using drugs and are connected to drug dealers; people of the second type are drug despisers: they find drug use repugnant and so they do not use drugs and if they observe someone seeking to buy drugs, with probability r they will report to the police and the police will act; the third type consists of drug neutrals who do not use drugs and are not aware of any source of drugs, but do not report drug related activities to the police. At any time $t = 0, 1, 2, ...$ denote by X_t, Y_t, Z_t the current number of drug users, despisers and neutrals in the system, respectively. With a mild abuse of notation we say a person belongs to X_t if he is a drug user, similarly for Y_t and Z_t.

At time $t = 0$ the population composition is (X_0, Y_0, Z_0) and at each time t one outsider joins the system. This outsider is either a drug despiser (with probability p) or a potential drug user (with probability $1 - p$). If he is a drug despiser, then he joins Y_t directly; if he is not a drug despiser, he needs to decide whether he should try to find drugs. He has two options: he could choose to live a peaceful life and join Z_t directly, or he could randomly draw a person from the current population, and ask: "do you know where I can find drugs?" If he asks this question to a current drug user, he will be introduced to a reliable drug dealer, receive drugs and join X_t. If he asks a member of Y_t, there is a probability r that he is reported to the police and is arrested, convicted, and punished; and with probability $1 - r$, the drug despiser will say "I don't know" (or the police will not act on the report), in which case this newcomer will draw another person (memorylessly) in the system and repeat this process. If he asks this question to a person in Z_t, he always receives the answer "I

currency regulations.

don't know," and he will again draw another person (memorylessly) in the system and repeat this process. If someone is caught and punished during the process of finding drugs, he later joins Z_t.

The utility to a potential drug user of getting drugs is normalized to 1, his utility of joining Z_t is 0, and his utility of going to jail is $-K$ for some $K > 0$. Denote by q the probability that he eventually finds drugs, if he decides to try. The easiest way to compute q is by first step analysis: in his first encounter, with probability $\frac{X_t}{X_t+Y_t+Z_t}$, he meets a drug user and successfully finds drugs. With probability $\frac{Y_t}{X_t+Y_t+Z_t}$, he meets a drug despiser and, conditional on that, with probability r he will be reported to the police and penalized, while with probability $1-r$, he needs to draw another person and his future probability of success is again q. With probability $\frac{Z_t}{X_t+Y_t+Z_t}$, he meets a drug neutral, and he will redraw and his future probability of success is q. Therefore

$$q = \frac{X_t}{X_t + Y_t + Z_t} \cdot 1 + \frac{Y_t}{X_t + Y_t + Z_t} \cdot r \cdot 0 + \frac{Y_t}{X_t + Y_t + Z_t} \cdot (1-r) \cdot q + \frac{Z_t}{X_t + Y_t + Z_t} \cdot q.$$

Solving for q we have $q = \frac{X_t}{X_t + r \cdot Y_t}$, and the probability of getting caught during the process is $1 - q = \frac{r \cdot Y_t}{X_t + r \cdot Y_t}$.

Then the newcomer should choose to enter the market and attempt to buy drugs if and only if his expected utility $1 \cdot \frac{X_t}{X_t + r \cdot Y_t} - K \cdot \frac{r \cdot Y_t}{X_t + r \cdot Y_t} > 0$, which simplifies to $X_t > Kr \cdot Y_t$.

This describes a Markov Chain and we are interested in how X_t, Y_t, Z_t evolve with time. Notice Z_t does not influence the newcomer's decision, therefore it does not matter when a prisoner is released from jail, as long as he joins Z_t afterwards. For simplicity, assume all prisoners are released during the same time period they join the system, so that we can describe the transition of the Markov Chain simply, as follows:[8]

$$\begin{aligned}
P\left((x+1,y,z)|(x,y,z)\right) &= (1-p)\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}}, \\
P\left((x,y+1,z)|(x,y,z)\right) &= p, \\
P\left((x,y,z+1)|(x,y,z)\right) &= (1-p)\frac{ry}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} + (1-p)\mathbf{1}_{\{\frac{x}{y}\leq Kr\}}.
\end{aligned}$$

There are three parameters that we take as fixed in this model that, when the results are interpreted can be viewed as responsive to policy decisions. The probability p that the newcomer finds drugs repugnant is something that a policy maker could seek to influence through education. The probability r that concerned citizens will report drug activity to the police and that the police and courts will act effectively on such reports could be influenced both by public relations and by changing the intensity of police activities. The size of the legal penalty K can be influenced by laws concerning the length of prison sentences or monetary penalties. However these may not all be easy to change, and in a more complete model these parameters could be at least partially endogenous. That is, the degree of public repugnance, and the willingness of police and juries and legislators to act against an illegal market may depend in part on how common are the illegal transactions and how large is the proposed punishment. In Appendix D we provide some simulations in this regard and show that our main results are robust.

A market becomes extinct if and only if the long run proportion of drug users in the population goes to 0. That is, we can seek to understand the probability of this event:

[8] The indicator functions require us to study not only the limit, but also the whole trajectory.

Definition 2.1. ***Market Extinction***:

$$\textit{Extinction} \quad = \quad \left\{ \lim_{t\to\infty} \frac{X_t}{X_t + Y_t + Z_t} = 0 \right\}.$$

Notice that if ever $X_t \leq Kr \cdot Y_t$, then for all $t' \geq t$, $X_{t'} \leq Kr \cdot Y_{t'}$, i.e. no newcomer after time t will try to find drugs. So if $X_t \leq Kr \cdot Y_t$ happens at some time t, then the market becomes extinct.

Hereafter we assume $X_0 > Kr \cdot Y_0$, so at least the first few newcomers will be attempting to acquire drugs.

Definition 2.2. *We define a stopping time:*

$$\tau = \inf \{t \in \mathbb{N} | X_t \leq Kr \cdot Y_t\},$$

and denote the ratio between drug users and drug despisers to be

$$R_t = \frac{X_t}{Y_t},$$

then an equivalent definition of τ *is*

$$\tau = \inf \{t \in \mathbb{N} | R_t \leq Kr\}.$$

We are interested in the following questions:

- Under which condition does the limit of $\frac{X_t}{X_t+Y_t+Z_t}$ go to 0, i.e. the market becomes extinct?
- If there is no extinction (we say the market **survives** in this case[9]), then what will be the long run composition of the market? In other words, will $(\frac{X_t}{X_t+Y_t+Z_t}, \frac{Y_t}{X_t+Y_t+Z_t}, \frac{Z_t}{X_t+Y_t+Z_t})$ have a limit?
- Suppose we are in a world in which the market could either survive or become extinct, then what do we know about the probability of eventual extinction?

We will answer the first two questions in section 3, and the last one in section 4. Before we carry out our analysis, here are two simple facts about the model:

Proposition 2.3. *Almost surely,* $\lim_{t\to\infty} \frac{Y_t}{X_t+Y_t+Z_t} = p$.

This follows from the strong law of large numbers. Note that the existence of $\lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t}$ does not follow from the strong law of large numbers, since the probability the newcomer decides to join X_t depends on the current state of the world.

Proposition 2.4. *If* $\tau < \infty$*, then almost surely,* $\lim_{t\to\infty}(\frac{X_t}{X_t+Y_t+Z_t}, \frac{Y_t}{X_t+Y_t+Z_t}, \frac{Z_t}{X_t+Y_t+Z_t}) = (0, p, 1-p)$*, therefore* $\tau = \infty$ *is a necessary condition for the market to survive.*[10]

[9] That is, the market survives if the limit of $\frac{X_t}{X_t+Y_t+Z_t}$ does not exist, or if the limit is non-zero.

[10] In fact, $\tau = \infty$ is also a sufficient condition for the market to survive. An informal argument goes as follows: Suppose $\tau = \infty$, then $\frac{X_t}{Y_t} > Kr$ for all t. By Proposition 2.3, $\frac{Y_t}{X_t+Y_t+Z_t} \stackrel{a.s.}{\to} p$. Therefore with probability 1, $\lim_{t\to\infty} \frac{X_t}{X_t+Y_t+Z_t} \neq 0$, i.e. the market survives. A formal proof of this statement can be found in section 3. We can then think of τ as the death time of the market, e.g. in part 2 of Theorem 4.1. In other words, the market becomes extinct if and only if entry eventually becomes unprofitable, and once it becomes unprofitable it stays unprofitable. This gives us another definition of extinction:

$$\text{Extinction} = \{\lim_{t\to\infty} R_t = 0\} = \{\lim_{t\to\infty} R_t < Kr\} = \{\tau < \infty\}.$$

(We haven't shown that $\lim_{t\to\infty} R_t$ exists when $\tau = \infty$, which is proven by Lemma B.1 in the appendix.)

This is simply a restatement of our analysis below Definition 2.1: once the stopping time is reached, X_t stays constant and its limiting proportion is zero.

Below is a summary of the model.

X_t	number of drug users in the system
Y_t	number of drug despisers in the system
Z_t	number of drug neutrals in the system
R_t	X_t/Y_t
p	probability that the newcomer is a drug despiser
r	probability of reporting to police (for Y_t)
1	utility of using drugs
0	utility of joining Z_t directly
$-K$	utility of getting caught
$q = \frac{X_t}{X_t + r \cdot Y_t}$	probability of finding drugs
τ	first time when $X_t \leq Kr \cdot Y_t$

Table 1: Summary of the model

3 Long Run Behavior

We first provide a heuristic analysis: suppose such a market survives and reaches a steady state, i.e. $\lim_{t\to\infty} R_t$ exists, then what should it be? By the law of large numbers we have:

$$\lim_{t\to\infty} \frac{X_t + Z_t}{Y_t} = \frac{1-p}{p}.$$

On the other hand, the long run ratio between X_t and Z_t should be the same as the ratio between the probability of the newcomer joining X_t and Z_t, therefore

$$\lim_{t\to\infty} \frac{X_t}{Z_t} = \lim_{t\to\infty} \frac{q}{1-q} = \lim_{t\to\infty} \frac{X_t}{r \cdot Y_t}.$$

Combine these two equations we obtain that

$$\lim_{t\to\infty} X_t : Y_t : Z_t = 1 - p - pr : p : pr.$$

Hence we should have

$$\boxed{\lim_{t\to\infty} R_t = \frac{1-p-pr}{p} \equiv \bar{R}}.$$

Note that $\bar{R}$ is independent of K, as conditional on market survival, the transition probabilities do not depend on K.

We can now compare this limiting ratio with the decision threshold Kr, and get three cases:

1. **Controllable:** $\bar{R} < Kr \Leftrightarrow p > \frac{1}{1+r+Kr}$. In this case, clearly R_t will eventually drop below Kr, which means the market can never survive.

2. **Borderline:** $\bar{R} = Kr \Leftrightarrow p = \frac{1}{1+r+Kr}$.

3. **Uncontrollable:** $\bar{R} > Kr \Leftrightarrow p < \frac{1}{1+r+Kr}$. Then conditional on surviving, the limit of R_t is indeed larger than Kr, so the market should have a chance to survive.

The difficult problem here is to show the existence of the limit of R_t. One may also be concerned that $\bar{R}$ could be negative depending on the parameter values, which is implausible (i.e. the quantity $\frac{1-p-pr}{p} \equiv \bar{R}$ may be negative, but $\lim_{t\to\infty} R_t$ can never be). We now formally state the first main theorem of this paper, which basically confirms this intuitive analysis.

Theorem 3.1 (**Controllable and uncontrollable black markets**). *There exist three cases,*

1. ***Controllable:*** $p(1+r+Kr) > 1$. *In this situation, the market will become extinct with probability 1 and* $R_t \overset{a.s.}{\to} 0$.

2. ***Borderline:*** $p(1+r+Kr) = 1$. *If in addition,* $1-p \geq 2pr \Leftrightarrow K \geq 1$ *then it behaves like the controllable case; if* $1-p < 2pr \Leftrightarrow K < 1$*, then it behaves like the uncontrollable case.*

3. ***Uncontrollable:*** $p(1+r+Kr) < 1$. *The market survives with positive probability. And when it survives,* $R_t \overset{a.s.}{\to} \bar{R}$.

Remark 3.2 (**Measuring repugnance**). *We can think of* $I \equiv p(1+r)$ *as an index reflecting the repugnance with which the market is perceived.* p *reflects the extent of repugnance (the proportion of people who find the market repugnant) and* r *represents the intensity of repugnance as reflected in the likelihood that a disappovingly-observed repugnant action will be reported and acted upon. When* $I > 1$*, the market is always controllable. When* $I = 1$*, the black market will also eventually become extinct no matter how small the punishment* K *is (note that* $\bar{R} = \frac{1-I}{p}$*).* $I < 1$ *means the general population finds the market insufficiently repugnant to guarantee that the black market will be controllable, and to make the market controllable the policy maker needs to be able to set a sufficiently large punishment. (The present simple model does not consider what limits might exist on how large a punishment can be set, or whether demanding too large a punishment would reduce the probability* r *that violations are reported and acted upon. See for example: Acemoglu and Jackson [2], and Akerlof and Yellen [3].)*

The proof of Theorem 3.1 can be found in Appendix B. Although the heuristic analysis appears to be simple and intuitive, the rigorous proof turns out to be highly non-trivial, especially for the borderline case. We end up borrowing a technique from population models in mathematical biology and study our discrete process through a continuous time Markov jump model.

The heuristic analysis above already explains why the market can never survive in the controllable case. Below we offer an argument on why the market survives with a positive probability in the uncontrollable case.

First we know that a necessary condition for market survival is $\tau = \infty$ by Proposition 2.4. On the other hand, it will also be sufficient: Lemma B.1 in the appendix states that $\tau = \infty$ implies $R_t = \frac{X_t}{Y_t} \overset{a.s.}{\to} \bar{R}$, and by Proposition 2.3, $\frac{Y_t}{X_t+Y_t+Z_t} \overset{a.s.}{\to} p$. Together they imply $\frac{X_t}{X_t+Y_t+Z_t} \overset{a.s.}{\to} p\bar{R} > pKr > 0$, which means the market survives with probability 1. Therefore our job is to show $\mathbb{P}[\tau = \infty] > 0$.

To analyze this problem, let's define $S_t = X_t - KrY_t$. Our initial condition implies that $S_0 > 0$, and it is clear that $X_t > Kr \cdot Y_t$ if and only if $S_t > 0$. Therefore the probability that a market survives equals to $\mathbb{P}[S_t > 0\ \forall t | S_0 > 0]$. How would S_t behave? As long as $S_t > 0$, then $S_{t+1} = S_t + 1$ with probability

$$q(1-p) = \frac{X_t}{X_t + r \cdot Y_t}(1-p) > \frac{K}{1+K}(1-p)$$

(since $S_t > 0 \Rightarrow Y_t < \frac{1}{Kr} \cdot X_t$); $S_{t+1} = S_t - Kr$ with probability p; and $S_{t+1} = S_t$ with probability

$$(1-q)(1-p) = \frac{r \cdot Y_t}{X_t + r \cdot Y_t}(1-p) < \frac{1}{1+K}(1-p).$$

We then define a new Markov chain $\bar{S}_t$: for $t \geq 1$ let

$$W_t = \begin{cases} 1 \text{ w.p. } \frac{K}{1+K}(1-p) \\ -Kr \text{ w.p. } p \\ 0 \text{ w.p. } \frac{1}{1+K}(1-p) \end{cases}$$

(w.p. stands for "with probability"), then define $\bar{S}_n = S_0 + \sum_{t=1}^{n} W_t$. It is clear that the probability that S_t never enters $(-\infty, 0]$ is lower bounded by that of $\bar{S}_t$. We know $\bar{S}_t$ is a random walk and notice that the drift

$$E(W_t) = 1 \times \frac{K}{1+K}(1-p) - Kr \times p = \frac{[1 - p(1 + r + Kr)]K}{1+K} > 0$$

when $p < \frac{1}{1+r+Kr}$, which means $\mathbb{P}[\bar{S}_t > 0 \ \forall t | S_0 > 0] > 0$, therefore $\mathbb{P}[S_t > 0 \ \forall t | S_0 > 0] > 0$, i.e. with a positive probability the market survives. This argument can be formalized through coupling, which can be found in the appendix. □

We will study this probability of market survival $\mathbb{P}[S_t > 0 \ \forall t | S_0 > 0]$ in detail in the next section.

Note that the comparative statics at the threshold $p(1 + r + Kr) = 1$ are clear: as p, r and K increase, it becomes easier to extinguish the market. However if p and K are not too large, then even the maximum intensity of repugnance, $r = 1$ may be insufficient to make the black market controllable.

One natural way of extending this model is by endogenizing the parameters p and r. In particular, the extent and intensity of repugnance may depend on the current proportion of drug despisers in the system. That is, $p_{t+1} = H(\frac{Y_t}{X_t+Y_t+Z_t})$ and $r_{t+1} = Q(\frac{Y_t}{X_t+Y_t+Z_t})$, where H and Q are two continuous functions. This extension is discussed in detail in Appendix D. Here we briefly present a few interesting features of this extension. First, p_t and r_t converge almost surely (as random variables), moreover, the limiting points of p_t, p^* satisfies $p^* = H(p^*)$; however not all the fixed points of H are stable: only some of them are in the support of the limit, while others are saddle points. In other words, the limit of p_t is a distribution over some fixed points of H. And the limit of r_t is $r^* = Q(p^*)$. Second, if some of these stabilizing p^*'s satisfy $p^*(1 + r^* + Kr^*) < 1$, then the market will survive with a positive probability, and $\bar{R}^* = \frac{1-p^*-p^*r^*}{p^*}$ is one potential limit for $R_t = \frac{X_t}{Y_t}$. (If multiple such $\bar{R}^*$'s exist, then the limit of R_t is a distribution over them.) Otherwise, if $p^*(1 + r^* + Kr^*) > 1$ for all such p^*, then the market becomes extinct almost surely. Simply put, the market isn't too different from the baseline model, we just replace p and r with p^* and r^*.[11]

Nonetheless, the world is no longer binary: instead of having only two possible limiting states, market survival and extinction, we may have different levels of drug activities when the market survives (i.e. when there are multiple $\bar{R}^*$'s). This is, in spirit, similar to the multiple equilibria discussions in the traditional economic models on crimes, although the underlying force here is no longer the strategic interactions, but the stochastic nature of the process. (See for example Glaeser et al [27] and Calvo-Armengol and Zenou [12].)

[11] We can formally prove the convergence of $\frac{Y_t}{X_t+Y_t+Z_t}$, p_t and r_t, but not the convergence of $\frac{X_t}{X_t+Y_t+Z_t}$, which is verified through simulations.

4 Extinction Probability and Speed

One might be surprised that the results in Theorem 3.1 have nothing to do with the initial state of the world (X_0, Y_0, Z_0), other than the assumption $X_0 > Kr \cdot Y_0$. It seems reasonable that a market which is infested with drug users would require more effort to eliminate. Indeed, in this section we show that in the uncontrollable case, i.e. when $p < \frac{1}{1+r+Kr}$, the probability of market extinction decays exponentially in the initial state of the world. (We are back to the constant p and r case.)

Before beginning the analysis, consider how policy makers could have some control over the initial states. One example would be the regulation of synthetic drugs. When a new synthetic drug becomes available, it takes time before it can be banned. The number of users it attracts before it is banned may be an important factor for the prospects of extinguishing the market. So the speed of initial regulation may be consequential, and there may be markets that could be successfully prevented only by prompt action, and not when they have become well established.

The exact probability of market survival, $\mathbb{P}[S_t > 0 \ \forall t | S_0 > 0]$ is quite difficult to compute directly. We will use $\mathbb{P}[\bar{S}_t > 0 \ \forall t | S_0 > 0]$ to provide a lower bound. The main technique we use here is the so called exponential martingale (Wald [49], see also chapter 7.5 of Gallager [26]). For the sake of exposition, we will define another stopping time:

$$\bar{\tau} = \inf \left\{ t \in \mathbb{N} | \bar{S}_t \leq 0 \right\}.$$

Then $\mathbb{P}[\bar{S}_t > 0 \ \forall t | S_0 > 0] = \mathbb{P}[\bar{\tau} = \infty]$.

Now we present the second main theorem of this paper:

Theorem 4.1 (The probability of market survival). *In the uncontrollable case:*

1. *The probability of market survival*

$$\mathbb{P}[S_t > 0 \ \forall t | S_0 > 0] \geq 1 - e^{\theta^* S_0},$$

where $S_0 = X_0 - Kr \cdot Y_0$, and θ^* is a negative constant.

2. *Conditional on market extinction, it decays exponentially fast. That is, for every $\theta \in (\theta^*, 0)$, and $t > 0$, we have*

$$\mathbb{P}[\tau > t | \tau < \infty] \leq \frac{1}{\mathbb{P}[\tau < \infty]} e^{\theta S_0 + \psi(\theta) t},$$

where $\psi(\theta) < 0$.

3. *Following the same notation as 1 and 2,*

$$\mathbb{P}[\tau < \infty | \tau > t] \leq \frac{e^{\theta S_0 + \psi(\theta) t}}{1 - e^{\theta^* S_0}}.$$

The first part says that the probability of market extinction decays exponentially in S_0. The second part says that if a market eventually becomes extinct, then the probability that it survives longer than t decays exponentially in t, for any given parameter values and initial states (X_0, Y_0, Z_0) (then $\mathbb{P}[\tau < \infty]$ is also fixed). And the third part says that the probability that the market eventually becomes extinct, decays exponentially in its current survival time t.

Below we provide a formal proof for the first part, so the readers can understand where these θ and ψ come from. The proof for part 2 is similar, which can be found in Appendix C. Part 3 follows straightforwardly from part 1 and 2.

Proof of part 1. Let's define (recall that $W_n = \bar{S}_n - \bar{S}_{n-1}$)

$$\phi(\theta) = \mathbb{E}[e^{\theta W_n}],$$

$$\psi(\theta) = \log(\phi(\theta)),$$

and

$$M_n(\theta) = e^{\theta \bar{S}_n - n\psi(\theta)}.$$

Then we can easily check that $M_n(\theta)$ is a martingale for any parameter value θ. In fact,

$$\mathbb{E}[M_{n+1}(\theta)|\bar{S}_1, \bar{S}_2, ..., \bar{S}_n] = e^{\theta \bar{S}_n}\mathbb{E}[e^{\theta W_{n+1}}|\bar{S}_1, \bar{S}_2, ..., \bar{S}_n]e^{-(n+1)\psi(\theta)} = M_n(\theta).$$

Next we show that the equation $\psi(\theta) = 0$, i.e. $\phi(\theta) = 1$ has two solutions. Notice $\phi(0) = 1$, so $\theta = 0$ is one of them. We can also compute that,

$$\phi''(\theta) = \mathbb{E}[W_n^2 e^{\theta W_n}] > 0$$

for all θ, i.e. ϕ is convex. Recall that in the uncontrollable case,

$$\phi'(0) = \mathbb{E}[W_n e^{0 \times W_n}] = \mathbb{E}[W_n] > 0.$$

Then for small enough ϵ, $\phi(-\epsilon) < 1$, on the other hand,

$$\lim_{\theta \to -\infty} \phi(\theta) > \lim_{\theta \to -\infty} p e^{-Kr\theta} = \infty.$$

Thus $\phi(\theta) = 1$ must have another root $\theta^* < 0$ by the intermediate value theorem. (And convexity of ϕ implies $\phi(\theta) = 1$ has no more than two roots, so $\phi(\theta) = 1$ has exactly two roots, 0 and θ^*.) Now we have $M_n(\theta^*) = e^{\theta^* \bar{S}_n}$ is a martingale. And we would like to apply the optional sampling theorem to it, with stopping time $\bar{\tau}$. However, the optional sampling theorem can not be directly applied to stopping times that are potentially unbounded (without further restrictions on the martingale), therefore we define another stopping time $\bar{\tau} \wedge t = \min\{\bar{\tau}, t\}$ for any finite time t, then by the optional sampling theorem, we have:

$$\mathbb{E}[e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}}] = e^{\theta^* S_0}$$

$$\Rightarrow \mathbb{E}[e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}}|\bar{S}_{\bar{\tau}\wedge t} \le 0]\mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} \le 0] + \mathbb{E}[e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}}|\bar{S}_{\bar{\tau}\wedge t} > 0]\mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} > 0] = e^{\theta^* S_0}.$$

Since $\theta^* < 0$, then

$$e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}} \ge e^{\theta^* \times 0} = 1$$

when $\bar{S}_{\bar{\tau}\wedge t} \le 0$, and notice

$$\mathbb{E}[e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}}|\bar{S}_{\bar{\tau}\wedge t} > 0]\mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} > 0] \ge 0.$$

Therefore we have

$$e^{\theta^* S_0} \ge \mathbb{E}[e^{\theta^* \bar{S}_{\bar{\tau}\wedge t}}|\bar{S}_{\bar{\tau}\wedge t} \le 0]\mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} \le 0] \ge \mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} \le 0].$$

By definition of $\bar{\tau}$, $\bar{S}_{\bar{\tau}\wedge t} \le 0$ if and only if $\bar{\tau} \le t$, then

$$\mathbb{P}[\bar{S}_{\bar{\tau}\wedge t} \le 0] = \mathbb{P}[\bar{\tau} \le t].$$

Thus

$$\mathbb{P}[\bar{\tau} \le t] \le e^{\theta^* S_0}.$$

Finally let $t \to \infty$, then

$$\mathbb{P}[\bar{\tau} < \infty] \le e^{\theta^* S_0}.$$

Therefore

$$\mathbb{P}[S_t > 0 \ \forall t | S_0 > 0] \ge \mathbb{P}[\bar{S}_t > 0 \ \forall t | S_0 > 0] = \mathbb{P}[\bar{\tau} = \infty] \ge 1 - e^{\theta^* S_0}.$$

□

Proof of part 3. The quantity $\mathbb{P}[\tau < \infty | \tau > t]$ can be calculated by the Bayesian formula

$$\mathbb{P}[\tau < \infty | \tau > t] = \frac{\mathbb{P}[\tau > t | \tau < \infty]\mathbb{P}[\tau < \infty]}{\mathbb{P}[\tau > t]}.$$

By 1 and 2 we have

$$\mathbb{P}[\tau > t] \geq \mathbb{P}[S_t > 0, \forall t | S_0 > 0] \geq 1 - e^{\theta^* S_0},$$
$$\mathbb{P}[\tau > t | \tau < \infty]\mathbb{P}[\tau < \infty] \leq e^{\theta S_0 + \psi(\theta) t}.$$

Then we conclude that

$$\mathbb{P}[\tau < \infty | \tau > t] \leq \frac{e^{\theta S_0 + \psi(\theta) t}}{1 - e^{\theta^* S_0}}.$$

□

Remark 4.2. *Part 3 of Theorem 4.1 tells us that if a market has survived for a very long time, then it is likely to survive forever. This implies that, if we struggle to kill a market for a long time without much success, then unless we can change the parameters, our chances of eventual success diminish rapidly.*

5 Discussion

Simple conceptual models like the one presented here are not meant to be simple guides to public policy, nor sources of precise prediction about particular markets. Policy decisions regarding specific markets require input from detailed studies of how each such market operates and responds to changes. The model presented here is intended rather to provide conceptual clarity to complex issues that may apply to many markets, and to provide some input of this kind to policy and design decisions.

Thus the model and its main results about the extent and intensity of repugnance, and the likelihood of extinguishing relatively well-established black markets (Theorem 3.1 and 4.1) can help us understand why some black markets persist but others do not, and when we might usefully consider harm reduction measures rather than simply pursuing the goal of driving the market to extinction.

For example, it appears that in California, where it is a felony to sell horse meat for human consumption, there is virtually no black market for horse meat.[12] In terms of our model, the reason is likely that restaurants that contemplate serving horse meat, ranchers and butchers who might like to supply it, and consumers who might like to eat it are deterred by the low potential reward (horsemeat may be tasty but it apparently isn't addictive), compared to the probability of detection and punishment. So it appears that this market is naturally controllable.

The case of prostitution is quite different: there are markets for prostitutes around the world, including in places like the U.S. where both sides of the transaction are illegal. However the maximum punishments prescribed by U.S. state laws are mild (compared for example to the punishments prescribed for drug offenses, and compared to other sex offences that require those convicted to

[12] See Roth [42] for background, and note that internet sites such as http://www.grubstreet.com/2013/03/20-restaurants-where-you-can-eat-horse-around-the-world.html, which list venues at which horse meat may be available in Toronto and Mexico and in some American states, have no listings for California.

register as sex offenders).[13] Indeed, the relatively low punishments (and infrequent enforcement, and legalization in many countries) may have evolved as harm reduction measures in the face of a historical inability to control this market even with larger punishments, and a judgment that e.g. filling the prisons with offenders might do more harm than good. Theorem 4.1 suggests that, given that a market of non-negligible size presently exists, it would now be very hard to extinguish it merely by increasing punishments to higher but historically ineffective levels.

The market for illegal narcotics is still different: as already noted, it persists despite harsh punishments. These punishments are apparently insufficient to deter new users, some of whom may perceive compellingly high rewards, having already become addicted via legally available drugs.[14] Because there is presently a big population of drug buyers and sellers, Theorem 4.1 suggests that continued strict enforcement of existing laws is unlikely to extinguish the market (but see Rosenstein [41] who reached the opposite conclusion from a position of great authority in law enforcement).

Finally, to return to the example mentioned in the introduction, it does not appear that any harm reduction measures are needed in connection with the spot market for hitmen in the United States. This market may be so widely and intensely viewed as repugnant so as to be naturally controllable, and even if not, the present apparently low population of buyers and sellers suggests that it is and can continue to be controlled with the feasibly large penalties that are already in place. The situation may be different in places with a substantially higher incidence of lethal violence.[15]

6 Conclusion

Designing legal marketplaces involves trying to make them safe and reliable enough to attract many participants. In the opposite direction, the idea behind laws that criminalize markets that some influential part of society finds repugnant is that the risk of being penalized will make the market unsafe, and deter participation.[16] However, if the market is insufficiently repugnant in extent or intensity, even substantial legal penalties "on the books" may be insufficient to deter participation if those penalties cannot gain enough social support to be reliably enforced. Note also that if the feasible punishment is not too large, and if the extent of repugnance among the population is low, then even the maximum intensity of repugnance among those who wish to ban the market may be insufficient to control the black market.[17] And as an illegal market becomes larger,

[13] For a list by state of penalties for prostitutes and for customers, see e.g. https://prostitution.procon.org/view.resource.php?resourceID=000119.

[14] While our simple model does not distinguish between different kinds of entrants to the market, there is increasing evidence that people who are already addicted to opioids legally prescribed for pain relief (see e.g. Finkelstein et al. [24]) often enter the market for illegal narcotics when their legal access is ended. See also Alpert et al. [1] who argue that the introduction of an abuse-deterrent version of OxyContin in 2010 increased heroin use, and Pitt, Humphreys, and Brandeau [37], who further argue that sharply restricting opioid prescriptions may be counterproductive, and that harm reduction measures will offer more immediate relief.

[15] See e.g. Shirk [47] and Dell [18] on drug violence in Mexico, and more recent news reports on murder rates in Brazil, Colombia, Mexico and Venezuala, such as https://www.cbc.ca/news/world/mexico-record-homicide-rate-1.4497466. While these reports do not distinguish between employed and spot market hitmen, in the context of our model the concern is that there would be some spillover as it becomes easier to find hitmen of any sort, see e.g. the report by Onishi and Gebrekidan [34] on political assassinations in South Africa.

[16] Illegal markets may also be unsafe because participants are deprived of the recourse to the laws that protect buyers and sellers in legal markets, and this may cause negative externalities in addition to the repugnance of the banned transactions themselves. For example, when heroin which may be mixed with fentanyl is purchased from criminals, there are few guarantees as to the purity and accuracy of the dose being purchased, which may lend itself to increased risk of fatal overdose. Harm reduction measures to reduce overdoses are sometimes opposed by those who regard the added risk as part of the deterrent to participation in the black market, i.e. as a feature of the market design, not an unwelcome side effect.

[17] So a small intense group may be sufficient to pass legislation, but insufficient to enforce it.

it becomes more likely that those who wish to participate in it can do so without encountering those who would penalize them. Consequently, black markets that have operated successfully for a long time become increasingly hard to eliminate if the underlying social parameters and legal punishments cannot be changed.

But changing social repugnance, and even increasing legal punishments in an effective way, may be difficult. Policy makers may be able to influence the extent or intensity of repugnance by education and public relations. But because legislators don't have easy or direct access to who feels how much repugnance, this is likely to be more difficult than passing legislation. At the very least, changing widespread attitudes takes time. And increasing mandated punishments beyond what social repugnance will support can be counterproductive if it makes citizens less likely to report illegal transactions and juries less likely to convict.[18] So we may never be able to completely eliminate some markets, despite the fact that they cause considerable harm. Hence harm reduction should be in our portfolio of design tools for dealing with repugnant markets that we can't extinguish despite the harm they may do.

Appendix A Notations and Preliminaries

We first introduce important definitions and lemmas used in the proof.

A.1 Markov Jump Process

We give a quick and intuitive introduction of Markov jump process. For details, consult Chapter 5 of Durrett [20] and Chapter 3 of Eberle [21] for example.

Definition A.1 (Markov jump process). *A Markov jump process is a random process* $\{\mathcal{U}_t\}_{\{t\geq 0\}}$ *taking values in a state space* E*. For every couple* $(i,j)\in E\times E, i\neq j$*, it has a rate of jump* q_{ij}*. Conditional on* $\mathcal{U}_t = i$*, we have a family of independent exponential random variables* τ_{ij} *of parameter* q_{ij}*. Then* $\mathcal{U}_t$ *stays constant for time* $\min_{j\in E}\tau_{ij}$ *and jumps to the new state:* $\arg\min_{j\in E}\tau_{ij}$*.*

Remark A.2. *This type of random process with "left limit and right continuous" is called "càdlàg" ("continue à droite et limite à gauch" in French).*

We can easily verify that it is a Markov process with the help of the memoryless property of exponential random variables. One equivalent definition gives the connection between Markov jump processes and classical discrete time Markov chains.

Definition A.3 (Equivalent definition). *Another way to define a Markov jump process is setting a series of jump times* $\{\sigma_n\}_{n\geq 0}$ *where* $\sigma_0 = 0$ *and* $\{\sigma_{n+1}-\sigma_n\}_{n\geq 0}$ *are independent exponential random variables of parameter* $\sum_{j\in E} q_{ij}$ *if* $\mathcal{U}_{\sigma_n} = i$*. At jump times, it's a Markov chain of transition matrix*

$$\mathbb{P}[\mathcal{U}_{\sigma_{n+1}} = j|\mathcal{U}_{\sigma_n} = i] = p_{ij} = \frac{q_{ij}}{\sum_{k\in E} q_{ik}}.$$

Corollary A.4 (Embedding). *If we only look at the process at jump times* $\{\mathcal{U}_{\sigma_n}\}_{n\geq 0}$*, it is a Markov chain of transition matrix* $p_{ij} = \frac{q_{ij}}{\sum_{k\in E} q_{ik}}$*.*

[18] See e.g. Bindler and Hjalmarsson [10], who point to an increase in convictions following a reduction in penalties.

A.2 Generator

The so-called generator is a useful tool in studying Markov jump processes. See Chapter 7 of Revuz and Yor [39] for details.

Definition A.5 (Generator). *Given a Markov jump process $\{\mathcal{U}_t\}_{\{t\geq 0\}}$ with jump rates $\{q_{ij}\}_{E\times E}$ and any function $f : E \to \mathbb{R}$, we define a generator $\mathcal{L}$ to be:*

$$\mathcal{L}f(i) = \sum_{j\in E} q_{ij}(f(j) - f(i)).$$

Proposition A.6. *If the expectation is well-defined, we have:*

$$\mathbb{E}[f(\mathcal{U}_t)] = \mathbb{E}\left[f(\mathcal{U}_0) + \int_0^t \mathcal{L}f(\mathcal{U}_s)\, ds\right] = \mathbb{E}\left[f(\mathcal{U}_0) + \int_0^t \sum_{j\in E} q_{\mathcal{U}_s j}(f(j) - f(\mathcal{U}_s))\, ds\right].$$

Moreover, $f(\mathcal{U}_t) - \int_0^t \mathcal{L}f(\mathcal{U}_s)\, ds$ defines a martingale.

A.3 Martingales and the Optional Sampling Theorem

We recall Doob's inequality, which is a useful tool in the analysis of càdlàg martingales.

Lemma A.7 (Doob's inequality). *[Le Gall [32], Proposition 3.8] Given $\{\mathcal{U}_t\}_{t\geq 0}$ a càdlàg super-martingale, then for all $t > 0, \lambda > 0$*

$$\lambda\mathbb{P}\left[\sup_{0\leq s\leq t} |\mathcal{U}_s| > \lambda\right] \leq \mathbb{E}[|\mathcal{U}_0|] + 2\mathbb{E}[|\mathcal{U}_t|],$$

in the case where $\{\mathcal{U}_t\}_{t\geq 0}$ is a martingale, we have

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |\mathcal{U}_s|^2\right] \leq 4\mathbb{E}[|\mathcal{U}_t|^2].$$

We mostly use the second inequality in this paper.

One of the most useful consequences of this lemma is the following martingale convergence theorem, readers are referred to Doob [19] for details.

Theorem A.8 (L^2 bounded martingale). *Given $\{\mathcal{U}_t\}_{t\geq 0}$ a family of L^2 bounded càdlàg sub-martingale (super-martingale) i.e.*

$$\sup_{t\geq 0} \mathbb{E}[\mathcal{U}_t^2] < \infty,$$

then there exists a limit $\mathcal{U}_\infty$ such that

$$\mathcal{U}_t \xrightarrow[a.s]{L^2} \mathcal{U}_\infty.$$

The optional sampling theorem, also called the optional stopping theorem, is a standard result in martingale theory.

Theorem A.9 (Optional sampling). *[Le Gall [32], Theorem 3.6] Let M_t be a martingale and T be a stopping time, then under certain conditions:*

$$E(M_T) = E(M_0).$$

One of the conditions that this result holds is when T is bounded almost surely, which is why we use $\bar{\tau} \wedge t$ for a finite t in the proof of Theorem 4.1. Another such condition is $\sup_{t\geq 0} \mathbb{E}[M_t^2] < \infty$ and $T < \infty$ almost surely. This version is used in proving Proposition B.3.

A.4 Coupling of Probability Spaces

Coupling is a very useful trick in statistics for comparing two random spaces in a deterministic way. To do this, we need to put many random spaces into one. This statement seems quite abstract, so we directly give its construction and then we shall see its advantages.

Definition A.10 (Canonical space)**.** *We construct a canonical random space* $(\Omega, \mathcal{F}, \mathbb{P})$ *as follows. Given a series of independent uniform* $[0,1]$ *random variables* $\{U_i\}_{i\geq 1}, \{V_i\}_{i\geq 1}$*, we construct* $(X_t, Y_t, Z_t)_{t\geq 0}$ *with initial data* (X_0, Y_0, Z_0)*:*

$$\begin{aligned} X_{t+1} &= X_t + \mathbf{1}_{\{p\leq U_{t+1}\leq 1\}}\mathbf{1}_{\{0\leq V_{t+1}<\frac{X_t}{X_t+rY_t}\}}, \\ Y_{t+1} &= Y_t + \mathbf{1}_{\{0\leq U_{t+1}<p\}}, \\ Z_{t+1} &= Z_t + \mathbf{1}_{\{p\leq U_{t+1}\leq 1\}}\mathbf{1}_{\{\frac{X_t}{X_t+rY_t}\leq V_{t+1}\leq 1\}}. \end{aligned}$$

One can directly verify that this construction agrees with our dynamics (before reaching τ). What's more, we could realize the dynamics of different parameters (p, r, K) in the same probability space so we could compare them path by path. In fact we have the following important lemma and will use it in the proofs.

Lemma A.11 (Monotonicity)**.** *In the canonical random space, we note the dynamics with parameter* p *by* $(X_t(p), Y_t(p), Z_t(p))_{t\geq 0}$ *and the slope by* $R_t(p)$*. Then* $\forall \omega \in \Omega$*,* $0 < p_1 < p_2$*, we have*

$$\begin{aligned} X_t(p_1)(\omega) &\geq X_t(p_2)(\omega), \\ Y_t(p_1)(\omega) &\leq Y_t(p_2)(\omega), \\ R_t(p_1)(\omega) &\geq R_t(p_2)(\omega). \end{aligned}$$

Proof. $Y_t(p_1)(\omega) \leq Y_t(p_2)(\omega)$ is very easy by observing $\mathbf{1}_{\{0\leq U_{t+1}<p_1\}(\omega)} \leq \mathbf{1}_{\{0\leq U_{t+1}<p_2\}(\omega)}$. The comparisons of X_t's and R_t's can be done by simple induction (together). □

Appendix B The Proofs in Section 3

In this subsection we prove Theorem 3.1. A key lemma of the proof is the following:

Lemma B.1 (Long run behavior). *Conditional on* $\tau = \infty$*,* $R_t \overset{a.s.}{\to} \bar{R} \vee 0$*. Concretely,*

1. ***Insufficient repugnance:*** *If* $p(1+r) < 1$*, then the limit of* R_t *is almost surely* $\bar{R}$*.*
2. ***Sufficient repugnance:*** *If* $p(1+r) \geq 1$*, then the limit of* R_t *is almost surely* 0*.*

To prove Lemma B.1, instead of studying our original discrete Markov process (X_t, Y_t, Z_t), we construct a new continuous process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}$[19] that reproduces the relevant properties of the discrete process, and use it to study their common behavior in the limit.

In continuous time, we consider a Markov jump process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}$:

1. It has the same initial state as the discrete process, i.e. $(\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0) = (X_0, Y_0, Z_0)$.

[19] The idea of studying a discrete process through a continuous time Markov jump model is first introduced in the work Athreya and Karlin [5] for the Pólya urn model.

2. At any moment, each drug user has two clocks which ring independently at exponential time with parameters $(1-p)$ and p respectively, and when the first rings a new drug user enters the market, while when the second rings a new drug despiser enters.

3. At any moment, each drug despiser has two clocks which ring independently at exponential time with parameters $(1-p)r$ and pr respectively, and when the first rings a new drug neutral enters the market, while when the second rings a new drug despiser enters.

4. The clocks of different individuals are all independent.

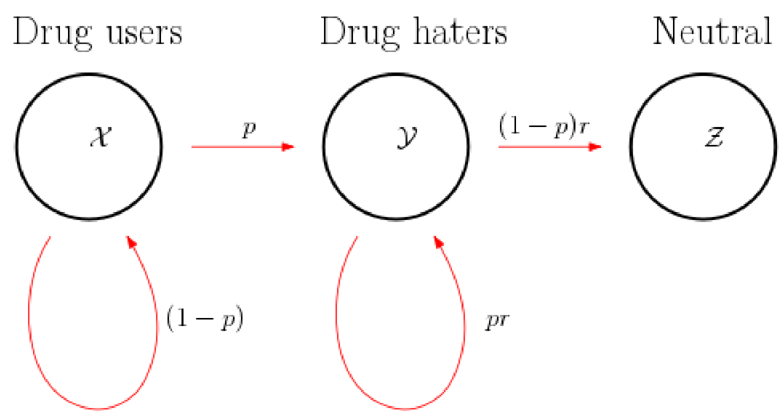

Figure 1: An image showing the (individual) rate of growth and mutation

Recall that for two independent exponential clocks Γ_1 and Γ_2 with rates μ_1 and μ_2, $\min\{\Gamma_1, \Gamma_2\}$ also follows an exponential distribution with rate $\mu_1 + \mu_2$, and $\mathbb{P}[\min\{\Gamma_1, \Gamma_2\} = \Gamma_1] = \frac{\mu_1}{\mu_1+\mu_2}$ and $\mathbb{P}[\min\{\Gamma_1, \Gamma_2\} = \Gamma_2] = \frac{\mu_2}{\mu_1+\mu_2}$.

Suppose we are at time t, with the current state of the world being $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)$, then the total rates of the next person joining $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are $(1-p)\mathcal{X}_t$, $p\mathcal{X}_t + pr\mathcal{Y}_t$ and $(1-p)r\mathcal{Y}_t$ respectively, and the total rate of the next arrival is the sum of all three, which is $\mathcal{X}_t + r\mathcal{Y}_t$. Therefore the probabilities that the next arrival joins $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are $(1-p)\frac{\mathcal{X}_t}{\mathcal{X}_t+r\mathcal{Y}_t}$, p and $(1-p)\frac{r\mathcal{Y}_t}{\mathcal{X}_t+r\mathcal{Y}_t}$ respectively, which agree with the transition probability of (X_t, Y_t, Z_t) before τ. The difference between these two processes is that, for the discrete process (X_t, Y_t, Z_t), the arrival time of the newcomer is always fixed at 1, while in the continuous process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)$, the arrival time of the newcomer follows an exponential distribution with parameter $\mathcal{X}_t + r\mathcal{Y}_t$. If we only document the states at the times of arrival in the continuous model, then it looks just like the discrete process. Indeed, let σ_i denote the time of i-th arrival, it is well-known that $(\mathcal{X}_{\sigma_i}, \mathcal{Y}_{\sigma_i}, \mathcal{Z}_{\sigma_i})$ has the same limiting behavior as (X_i, Y_i, Z_i), conditional on $\tau = \infty$. Readers are referred to Durrett [20] and Grimmett and Stirzaker [28] for details. In other words, we shall study the limit of $\mathcal{R}_t = \mathcal{X}_t/\mathcal{Y}_t$ which will be the same as the limit of R_t, conditional on $\tau = \infty$.

For the sake of exposition, let's define another random variable

$$\mathcal{W}_t \equiv \mathcal{Y}_t - (\bar{R})^{-1}\mathcal{X}_t$$

when $\bar{R} \neq 0$.

We begin by computing the first and second moments of $\mathcal{X}_t$ and $\mathcal{W}_t$.

1. **Expectation and second moment:** We start the proof by computing the expectation and second moment of $\mathcal{X}_t$. Using the formula provided by the generator (Proposition A.6) with

$f(x) = x$:[20]

$$\begin{aligned}\mathbb{E}[\mathcal{X}_t] &= \mathcal{X}_0 + \mathbb{E}\left[\int_0^t (1-p)\mathcal{X}_{s-}\, ds\right] \\ &\Rightarrow \boxed{\mathbb{E}[\mathcal{X}_t] = \mathcal{X}_0 e^{(1-p)t}}.\end{aligned}$$

We can also compute its second moment by using $f(x) = x^2$ in Proposition A.6: Since the jump rate at s is $(1-p)\mathcal{X}_{s-}$, we get

$$\begin{aligned}\mathbb{E}[\mathcal{X}_t^2] &= \mathcal{X}_0^2 + \mathbb{E}\left[\int_0^t (1-p)\mathcal{X}_{s-}((\mathcal{X}_{s-}+1)^2 - (\mathcal{X}_{s-})^2)\, ds\right] \\ &= \mathcal{X}_0^2 + \int_0^t 2(1-p)\mathbb{E}[\mathcal{X}_{s-}^2] + (1-p)\mathbb{E}[\mathcal{X}_{s-}]\, ds \\ &\Rightarrow \boxed{\mathbb{E}[\mathcal{X}_t^2] = \mathcal{X}_0^2 e^{2(1-p)t} + \mathcal{X}_0(e^{2(1-p)t} - e^{(1-p)t})}.\end{aligned}$$

However, since there is immigration from $\mathcal{X}$ to $\mathcal{Y}$, the expectation and second moment of $\mathcal{Y}$ are not easy to compute directly. Hence we study $\mathcal{W}_t$ instead. Similarly to before, we have

$$\begin{aligned}\mathbb{E}[\mathcal{W}_t] &= \mathcal{W}_0 + \mathbb{E}\left[\int_0^t p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-}) - (\bar{R})^{-1}(1-p)\mathcal{X}_{s-}\, ds\right] \\ &= \mathcal{W}_0 + \int_0^t pr\mathbb{E}[\mathcal{W}_s]\, ds \\ &\Rightarrow \boxed{\mathbb{E}[\mathcal{W}_t] = \mathcal{W}_0 e^{prt}}.\end{aligned}$$

The calculation of the second moment is more complicated:

$$\begin{aligned}\mathbb{E}[\mathcal{W}_t^2] &= \mathcal{W}_0^2 + \mathbb{E}\Big[\int_0^t p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-})\left((\mathcal{W}_{s-}+1)^2 - \mathcal{W}_{s-}^2\right) \\ &\quad +(1-p)\mathcal{X}_{s-}\left((\mathcal{W}_{s-} - (\bar{R})^{-1})^2 - \mathcal{W}_{s-}^2\right)\, ds\Big] \\ &= \mathcal{W}_0^2 + \mathbb{E}\Big[\int_0^t p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-})(2\mathcal{W}_{s-}+1) \\ &\quad +(1-p)\mathcal{X}_{s-}\left(-2(\bar{R})^{-1}\mathcal{W}_{s-} + (\bar{R})^{-2}\right)\, ds\Big] \\ &= \mathcal{W}_0^2 + \mathbb{E}\Bigg[\int_0^t 2\mathcal{W}_{s-}\underbrace{\left(p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-}) - (\bar{R})^{-1}(1-p)\mathcal{X}_{s-}\right)}_{=pr\mathcal{W}_{s-}} \\ &\quad +p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-}) + (\bar{R})^{-2}(1-p)\mathcal{X}_{s-}\, ds\Bigg] \\ &= \mathcal{W}_0^2 + \mathbb{E}\left[\int_0^t 2pr\mathcal{W}_{s-}^2 + p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-}) + (\bar{R})^{-2}(1-p)\mathcal{X}_{s-}\, ds\right].\end{aligned}$$

[20]We recall the explicit formula for first order differential equations:

$$\frac{d}{dt}f(t) = \gamma f(t) + g(t) \Longrightarrow f(t) = f(0)e^{\gamma t} + \int_0^t e^{\gamma(t-s)}g(s)\, ds.$$

Now solve for $\mathbb{E}[\mathcal{W}_t^2]$:

$$\begin{aligned}\mathbb{E}[\mathcal{W}_t^2] &= \mathcal{W}_0^2 e^{2prt} + \int_0^t e^{2pr(t-s)}\mathbb{E}\left[p(\mathcal{X}_{s-} + r\mathcal{Y}_{s-}) + (\bar{R})^{-2}(1-p)\mathcal{X}_{s-}\right] ds \\ &= \mathcal{W}_0^2 e^{2prt} + \int_0^t e^{2pr(t-s)}\mathbb{E}\left[pr\mathcal{W}_{s-} + \left(p + (\bar{R})^{-2}(1-p) + pr\bar{R}^{-1}\right)\mathcal{X}_{s-}\right] ds \\ &= \mathcal{W}_0^2 e^{2prt} + \int_0^t e^{2pr(t-s)}\left[pr\mathcal{W}_0 e^{prs} + \left(p + (\bar{R})^{-2}(1-p) + pr\bar{R}^{-1}\right)\mathcal{X}_0 e^{(1-p)s}\right] ds \\ &\Rightarrow \boxed{\mathbb{E}[\mathcal{W}_t^2] \leq \mathcal{W}_0^2 e^{2prt} + C_1(t)e^{(1-p)t} + C_2(t)e^{prt} + C_3(t)e^{2prt}}.\end{aligned}$$

Here we neglect the explicit expressions of C_1, C_2, C_3 but they are polynomials of t with degree at most 1.

2. **Insufficient repugnance case:** In this case, we have $1-p > pr$ (i.e. $\bar{R} > 0$) and we prove the following results:

Proposition B.2 (Scaling limit of $e^{-(1-p)t}\mathcal{X}_t$ and $e^{-(1-p)t}\mathcal{W}_t$)**.** *In the case $1-p > pr$,*

(a) $\{e^{-(1-p)t}\mathcal{X}_t\}_{t\geq 0}$ is a positive martingale which converges almost surely and in L^2 to a limit $\mathcal{E}$ that is positive almost surely.

(b) $\{e^{-prt}\mathcal{W}_t\}_{t\geq 0}$ is a martingale and $\{e^{-(1-p)t}\mathcal{W}_t\}_{t\geq 0}$ converges almost surely and in L^2 to 0.

Proof. Using the formula of expectations: $\forall 0 \leq s < t$,

$$\begin{aligned}\mathbb{E}[\mathcal{X}_t|\mathcal{F}_s] = \mathcal{X}_s e^{(1-p)(t-s)} &\;\Rightarrow\; \mathbb{E}[e^{-(1-p)t}\mathcal{X}_t|\mathcal{F}_s] = e^{-(1-p)s}\mathcal{X}_s, \\ \mathbb{E}[\mathcal{W}_t|\mathcal{F}_s] = \mathcal{W}_s e^{pr(t-s)} &\;\Rightarrow\; \mathbb{E}[e^{-prt}\mathcal{W}_t|\mathcal{F}_s] = e^{-prs}\mathcal{W}_s.\end{aligned}$$

So $\{e^{-(1-p)t}\mathcal{X}_t\}_{t\geq 0}$ and $\{e^{-prt}\mathcal{W}_t\}_{t\geq 0}$ are indeed martingales. Then, using the formula of the second moment, we have

$$\sup_{t\geq 0}\mathbb{E}[(e^{-(1-p)t}\mathcal{X}_t)^2] = \sup_{t\geq 0}\left(\mathcal{X}_0^2 + \mathcal{X}_0(1 - e^{-(1-p)t})\right) < \infty,$$

which implies the almost sure and L^2 convergence of $\{e^{-(1-p)t}\mathcal{X}_t\}_{t\geq 0}$ by Theorem A.8. Identifying the exact limit of $e^{-(1-p)t}\mathcal{X}_t$ is more complicated and has no direct use in our proof; we only need the fact that it is positive almost surely. In fact, it is known to be an exponential distribution, which serves our purpose.[21] We refer the readers to page 109 of Athreya and Ney [6], where we get the explicit formula for the quantity $\mathbb{E}[s^{\mathcal{X}_t}]$:

$$\mathbb{E}[s^{\mathcal{X}_t}] = \frac{se^{-(1-p)t}}{1 - (1 - e^{-(1-p)t})s}.$$

We take that $s = e^{he^{-(1-p)t}}, h < 1$, in this formula and let t go to ∞, then we get the limiting moment-generating function for $e^{-(1-p)t}\mathcal{X}_t$:

$$\mathbb{E}[e^{he^{-(1-p)t}\mathcal{X}_t}] = \frac{e^{he^{-(1-p)t}}e^{-(1-p)t}}{1 - (1 - e^{-(1-p)t})e^{he^{-(1-p)t}}} \xrightarrow{t\to\infty} \frac{1}{1-h}.$$

[21] This statement and the following verification assumes $\mathcal{X}_0 = 1$. When $\mathcal{X}_0 > 1$, the limiting distribution will be the sum of $\mathcal{X}_0$ many independent exponential distributions, which is a Gamma distribution $\Gamma(\mathcal{X}_0, 1)$.

This implies the limit follows a exponential distribution of parameter 1.

The treatment of $e^{-(1-p)t}\mathcal{W}_t$ is more difficult since $(1-p)$ is not the proper power to make it a martingale, while $e^{-prt}\mathcal{W}_t$ is not always bounded in L^2 (One can see this from the moments of $\mathcal{W}_t$). So we go back to Doob's inequality in Lemma A.7.

$$\begin{aligned}
& \mathbb{E}\left[\max_{n\leq t<n+1}|e^{-(1-p)t}\mathcal{W}_t|^2\right] \\
\leq\ & e^{-2(1-p-pr)n}\mathbb{E}\left[\max_{n\leq t<n+1}|e^{-prt}\mathcal{W}_t|^2\right] \\
\leq\ & e^{-2(1-p-pr)n}\left(4\mathbb{E}[(e^{-pr(n+1)}\mathcal{W}_{n+1})^2]\right) \\
\leq\ & 4e^{-2(1-p)n}\left(C_1'(n)e^{(1-p)n}+C_2'(n)e^{prn}+C_3'(n)e^{2prn}\right) \\
\leq\ & 4\left((C_2'(n)+C_3'(n))e^{-2(1-p-pr)n}+C_1'(n)e^{-(1-p)n}\right) \\
\to\ & 0. \\
\Rightarrow\ & \mathbb{E}[|e^{-(1-p)t}\mathcal{W}_t|^2]\leq \mathbb{E}\left[\max_{\lfloor t\rfloor\leq t<\lfloor t\rfloor+1}|e^{-(1-p)t}\mathcal{W}_t|^2\right]\xrightarrow{t\to\infty}0,
\end{aligned}$$

where C_1', C_2' and C_3' are polynomials of degree at most 1. This gives the L^2 convergence. Furthermore, by Markov's inequality we have:

$$\begin{aligned}
& \sum_{n=1}^{\infty}\mathbb{P}\left[\max_{n\leq t<n+1}|e^{-(1-p)t}\mathcal{W}_t|>\epsilon\right] \\
\leq\ & \sum_{n=1}^{\infty}\frac{1}{\epsilon^2}\mathbb{E}\left[\max_{n\leq t<n+1}|e^{-(1-p)t}\mathcal{W}_t|^2\right] \\
\leq\ & \sum_{n=1}^{\infty}\frac{1}{\epsilon^2}4\left((C_2'(n)+C_3'(n))e^{-2(1-p-pr)n}+C_1'(n)e^{-(1-p)n}\right) \\
<\ & \infty \\
\Rightarrow\ & \mathbb{P}\left[\left\{\max_{n\leq t<n+1}|e^{-(1-p)t}\mathcal{W}_t|>\epsilon\right\}\ i.o.\right]=0,
\end{aligned}$$

by the Borel-Cantelli lemma. This gives the desired almost sure convergence. □

Finally, we conclude that Proposition B.2 implies our result in the insufficient repugnance case, by the continuous mapping theorem:

$$\frac{\mathcal{X}_t}{\mathcal{Y}_t}=\frac{e^{-(1-p)t}\mathcal{X}_t}{e^{-(1-p)t}\mathcal{W}_t+(\bar{R})^{-1}e^{-(1-p)t}\mathcal{X}_t}\overset{a.s.}{\to}\bar{R}.$$

3. **Sufficient repugnance case:** $p(1+r)\geq 1\Leftrightarrow \bar{R}\leq 0\Leftrightarrow 1-p\leq pr$.

   We recall Lemma A.11 of monotonicity. Then $\forall 1\geq p\geq\frac{1}{1+r}$,

$$0\leq\lim_{t\to\infty}R_t(p)\leq\lim_{t\to\infty}R_t\left(\frac{1}{1+r}\right)\leq\lim_{q\nearrow\frac{1}{1+r}^-}\lim_{t\to\infty}R_t(q)=0,$$

   where $\lim_{t\to\infty}R_t(p)\geq 0$ comes from the fact that X_t, Y_t are positive. Therefore $R_t\overset{a.s.}{\to}0$ in the sufficient repugnance case, which concludes the proof of Lemma B.1. □

A direct consequence of Lemma B.1 is that, in the controllable case, $p(1+r+Kr) > 1$, i.e. $\bar{R} < Kr$, the market will become extinct with probability 1. Suppose otherwise, then $\tau = \infty$ by Proposition 2.4, and by Lemma B.1, R_t will eventually drop below Kr, which is a contradiction. This proves statement (1) of Theorem 3.1.

For the uncontrollable case, i.e. when $p < \frac{1}{1+r+Kr} \Leftrightarrow \bar{R} > Kr$, Lemma B.1 implies that if the market survives, then $R_t \overset{a.s.}{\to} \bar{R}$. Next we show that the probability of market extinction is strictly less than 1, with the help of the canonical space introduced in Definition A.10.

Proof. To study whether R_t will ever go below Kr, i.e. whether $\tau < \infty$, we study $S_t = X_t - KrY_t$ in the canonical space (it is clear from definition that $R_t \leq Kr \Leftrightarrow S_t \leq 0$):

$$S_{t+1} = S_t + \mathbf{1}_{\{p \leq U_{t+1} < 1\}} \mathbf{1}_{\{0 \leq V_{t+1} < \frac{X_t}{X_t + rY_t}\}} - Kr\mathbf{1}_{\{0 \leq U_{t+1} < p\}}.$$

It looks like a random walk. So we compare it with a simple random walk

$$\bar{S}_{t+1} = \bar{S}_t + \mathbf{1}_{\{p \leq U_{t+1} < 1\}} \mathbf{1}_{\{0 \leq V_{t+1} < \frac{K}{1+K}\}} - Kr\mathbf{1}_{\{0 \leq U_{t+1} < p\}}.$$

Before τ, we have $\frac{X_t}{X_t + rY_t} > \frac{K}{1+K}$, so

$$\mathbf{1}_{\{0 \leq V_{t+1} < \frac{X_t}{X_t + rY_t}\}}(\omega) \geq \mathbf{1}_{\{0 \leq V_{t+1} < \frac{K}{1+K}\}}(\omega).$$

By recurrence we obtain that $\forall 0 \leq t \leq \tau, S_t \geq \bar{S}_t$.

This is good news since we understand the behavior of random walks well. By computing the drift in the uncontrollable case:

$$\mathbb{E}[\bar{S}_{t+1} - \bar{S}_t] = \frac{(1-p)K}{1+K} - Krp = \frac{[1 - p(1+r+Kr)]K}{1+K} > 0,$$

thus $\{\bar{S}_t\}_{t \geq 0}$ has a positive probability to escape to infinity without ever touching the negative axis, so does $\{S_t\}_{t \geq 0}$ since it always stays right of the former.[22] (See chapter 4 of Durrett [20] for details.) This means there is a positive probability that $R_t > Kr,\ \forall t$, i.e. $\tau = \infty$. In other words, there is a positive probability that the market will survive, which concludes the proof of statement (3) of Theorem 3.1. □

Finally in the borderline case, the question is whether the process $(\mathcal{X}_t, \mathcal{Y}_t)$ will reach a state where $\mathcal{R}_t \leq \bar{R} = Kr$, or equivalently whether the process $\mathcal{W}_t$ will ever reach the positive axis (recall that $\mathcal{W}_0$ is assumed to be negative, when $\bar{R} = Kr$). If we define

$$T = \inf\{t | \mathcal{W}_t \geq 0\},$$

then the problem becomes whether $T < \infty$ almost surely. The answer depends on the parameters, since the size of the variance of $\mathcal{W}_t$ depends on them. More precisely, we can summarize the results in the following proposition:

Proposition B.3. *In the borderline case, i.e. when $\bar{R} = Kr$:*

[22] To be rigorous, we need to show that $S_t > 0,\ \forall t$ given that $\bar{S}_t > 0,\ \forall t$. We can prove it by induction. The base case is trivial: $S_0 = \bar{S}_0 > 0$. Suppose the statement is true at time t, i.e. $S_t > 0$, then $\tau > t$, and since t is discrete, we have $\tau \geq t+1$. Thus $S_{t+1} \geq \bar{S}_{t+1} > 0$, which finishes the inductive step. Therefore indeed $\tau = \infty$ and $S_t \geq \bar{S}_t,\ \forall t$.

1. ***Small variance:*** $1-p<2pr \iff K<1$. *Then T has a positive probability to be infinite and $(\mathcal{X}_t,\mathcal{Y}_t)$ has a positive probability to always stay above the slope $\bar{R}$, in other words, the market has a positive probability of surviving.*

2. ***Big variance:*** $1-p\geq 2pr \iff K\geq 1$. *Then T is almost surely finite and $(\mathcal{X}_t,\mathcal{Y}_t)$ will finally pass the slope, meaning the market always becomes extinct.*

To prove Proposition B.3, first we need to carefully compute the second moment of $\mathcal{W}_t$, continuing from:

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \int_0^t e^{2pr(t-s)} \left[pr\mathcal{W}_0 e^{prs} + \left(p+(\bar{R})^{-2}(1-p)+pr\bar{R}^{-1}\right)\mathcal{X}_0 e^{(1-p)s}\right] ds.$$

Denote $A=\left(p+(\bar{R})^{-2}(1-p)+pr\bar{R}^{-1}\right)$, then

- $1-p-2pr>0$**:**

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt}-e^{prt}) + \frac{A}{1-p-2pr}\mathcal{X}_0(e^{(1-p)t}-e^{2prt}),$$

  and the typical size of $\mathbb{E}[\mathcal{W}_t^2]$ is at the order of $e^{(1-p)t}$.

- $1-p-2pr=0$**:**

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt}-e^{prt}) + A\mathcal{X}_0 t e^{2prt},$$

  and the typical size is of te^{2prt}.

- $1-p-2pr<0$**:**

$$\mathbb{E}[\mathcal{W}_t^2] = \mathcal{W}_0^2 e^{2prt} + \mathcal{W}_0(e^{2prt}-e^{prt}) + \frac{A}{2pr-(1-p)}\mathcal{X}_0(e^{2prt}-e^{(1-p)t}).$$

  This expression is the same as in the first case, but its typical size is of e^{2prt}.

Now we are ready to prove the borderline case:

Proof of Proposition B.3.

1. **Small variance ($1-p<2pr$):** We prove by contradiction, suppose that $T<\infty$ almost surely, and we observe that (recall that $T=\inf\{t|\mathcal{W}_t\geq 0\}$):

$$\begin{aligned}\mathbb{E}[(e^{-prt}\mathcal{W}_t)^2] &= \mathcal{W}_0^2 + \mathcal{W}_0(1-e^{-prt}) + \frac{A}{2pr-(1-p)}\mathcal{X}_0(1-e^{(1-p-2pr)t}) \\ \Rightarrow &\ \sup_{t\geq 0}\mathbb{E}[(e^{-prt}\mathcal{W}_t)^2] < \infty.\end{aligned}$$

   So applying Theorem A.9 to the martingale $\{e^{-prt}\mathcal{W}_t\}_{t\geq 0}$ and we obtain

$$0 \leq \mathbb{E}[e^{-prT}\mathcal{W}_T] = \mathbb{E}[\mathcal{W}_0] < 0,$$

   which is a contradiction.

2. **Big variance ($1 - p \geq 2pr$):** We define

$$\mathcal{M}_t = e^{-prt}\mathcal{W}_t$$

and

$$T_{-M} = \inf\{t | \mathcal{M}_t \leq -M\}.$$

Notice now the definition of T is equivalent to

$$T = \inf\{t | \mathcal{M}_t \geq 0\},$$

and we define a condition ($\bigstar$):

$$\forall M > 0, T \wedge T_{-M} := \min\{T, T_{-M}\} < \infty \text{ a.s.}$$

Suppose that condition ($\bigstar$) is satisfied, then

$$\left(\mathcal{M}_t^{T \wedge T_{-M}}\right)_{t\geq 0} := \left(\mathcal{M}_{t\wedge T \wedge T_{-M}}\right)_{t\geq 0}$$

is a bounded martingale, so it has bounded L^2 norm and we could apply Theorem A.9 again and obtain that (notice $\mathcal{M}_T \leq 1, \mathcal{M}_{T_{-M}} \leq -M$):

$$\begin{aligned}
\mathcal{W}_0 &= \mathbb{E}[\mathcal{M}_0] = \mathbb{E}[\mathcal{M}_{T\wedge T_{-M}}] \\
&= \mathbb{E}\left[\mathcal{M}_T \mathbf{1}_{\{T<T_{-M}\}}\right] + \mathbb{E}\left[\mathcal{M}_{T_{-M}} \mathbf{1}_{\{T\geq T_{-M}\}}\right] \\
&\leq \mathbb{P}[T < T_{-M}] + (-M)(1 - \mathbb{P}[T < T_{-M}]) \\
&\Rightarrow M + \mathcal{W}_0 \leq (M+1)\mathbb{P}[T < T_{-M}] \\
&\Rightarrow \mathbb{P}[T < T_{-M}] \geq \frac{M + \mathcal{W}_0}{M+1}.
\end{aligned}$$

By passing M to ∞, we get

$$\mathbb{P}[T < \infty] = \lim_{M\to\infty} \mathbb{P}[T < T_{-M}] \geq \lim_{M\to\infty} \frac{M + \mathcal{W}_0}{M+1} = 1.$$

The rest is devoted to verifying condition ($\bigstar$). By Theorem A.8: since $(\mathcal{M}_{t\wedge T\wedge T_{-M}})_{t\geq 0}$ is a bounded martingale, it converges. This means either it touches the two barriers, or it converges without touching the two barriers. What we need to show is that the latter will not happen.

By way of contradiction, suppose ($\bigstar$) is not true, i.e. assume

$$\mathbb{P}[T \wedge T_{-M} = \infty] = \epsilon > 0.$$

Using the formula of the second moment of $\mathcal{W}_t$, we can compute that

$$\lim_{t\to s^+} \frac{\mathbb{E}[\mathcal{W}_t^2 | \mathcal{F}_s] - \mathcal{W}_s^2}{t-s} = 2pr\mathcal{W}_s^2 + pr\mathcal{W}_s + A\mathcal{X}_s.$$

Using the product rule for derivatives:

$$\lim_{t\to s^+}\frac{\mathbb{E}[e^{-2prt}\mathcal{W}_t^2|\mathcal{F}_s]-e^{-2prs}\mathcal{W}_s^2}{t-s} = -2pre^{-2prs}\mathcal{W}_s^2+e^{-2prs}(2pr\mathcal{W}_s^2+pr\mathcal{W}_s+A\mathcal{X}_s) = e^{-2prs}(pr\mathcal{W}_s+A\mathcal{X}_s).$$

Integrate for both sides we get

$$\mathbb{E}\left[e^{-2prt}\mathcal{W}_t^2-\int_s^t e^{-2pru}(pr\mathcal{W}_u+A\mathcal{X}_u)\,m(du)\,|\,\mathcal{F}_s\right]=e^{-2prs}\mathcal{W}_s^2,$$

where m denotes the usual Lebesgue measure.

This means $\left(e^{-2prt}\mathcal{W}_t^2-\int_0^t e^{-2pru}(pr\mathcal{W}_u+A\mathcal{X}_u)\,m(du)\right)_{t\geq 0}$ is a martingale, so is the version with the stopping time:

$$\left(e^{-2prt\wedge T\wedge T_{-M}}\mathcal{W}_{t\wedge T\wedge T_{-M}}^2-\int_0^{t\wedge T\wedge T_{-M}} e^{-2pru}(pr\mathcal{W}_u+A\mathcal{X}_u)\,m(du)\right)_{t\geq 0}.$$

Using the definition of $\left(\mathcal{M}_t^{T\wedge T_{-M}}\right)_{t\geq 0}$, we have:

$$\left((\mathcal{M}_t^{T\wedge T_{-M}})^2-\int_0^{t\wedge T\wedge T_{-M}}(pre^{-pru}\mathcal{M}_u+e^{-2pru}A\mathcal{X}_u)\,m(du)\right)_{t\geq 0}$$

is a martingale. Therefore,

$$\mathbb{E}\left[(\mathcal{M}_t^{T\wedge T_{-M}})^2-\int_0^{t\wedge T\wedge T_{-M}}(pre^{-pru}\mathcal{M}_u+e^{-2pru}A\mathcal{X}_u)\,m(du)\right]=\mathcal{M}_0^2.$$

Rearrange, we get:

$$\mathbb{E}\left[(\mathcal{M}_t^{T\wedge T_{-M}})^2-\int_0^{t\wedge T\wedge T_{-M}} pre^{-pru}\mathcal{M}_u\,m(du)\right]-\mathcal{M}_0^2=\mathbb{E}\left[\int_0^{t\wedge T\wedge T_{-M}} e^{-2pru}A\mathcal{X}_u\,m(du)\right]. \tag{B.4}$$

When $u\leq T\wedge T_{-M}$, $\mathcal{M}_u$ is bounded, so the integral in the LHS of eq. (B.4) is bounded. Also, $(\mathcal{M}_t^{T\wedge T_{-M}})_{t\geq 0}$ is bounded, therefore

$$\forall t>0, \text{LHS of eq. (B.4) is bounded}$$
$$\Rightarrow \limsup_{t\to\infty} \text{LHS of eq. (B.4) is bounded.}$$

On the other hand, by Proposition B.2, for almost every sample path ω, $e^{-(1-p)t}\mathcal{X}_t(\omega)$ converges to a positive limit $\mathcal{E}(\omega)>0$, thus $\exists N(\omega)>0$ such that $\forall u>N(\omega), e^{-(1-p)u}\mathcal{X}_u(\omega)\geq$

$\frac{1}{2}\mathcal{E}(\omega)$. Moreover, since $1-p \geq 2pr$, we have $e^{-2pru}\mathcal{X}_u \geq e^{-(1-p)u}\mathcal{X}_u$. Therefore,

$$\begin{aligned}
& \limsup_{t\to\infty} \int_0^t e^{-2pru} A\mathcal{X}_u(\omega)\mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\, m(du) \\
\geq & \limsup_{t\to\infty} \int_{N(\omega)}^t e^{-2pru} A\mathcal{X}_u(\omega)\mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\, m(du) \\
\geq & \limsup_{t\to\infty} \int_{N(\omega)}^t e^{-(1-p)u} A\mathcal{X}_u(\omega)\mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\, m(du) \\
\geq & \limsup_{t\to\infty} \int_{N(\omega)}^t \frac{1}{2}\mathcal{E}(\omega) A\mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\, m(du) \\
= & \infty \cdot \mathbf{1}_{\{T\wedge T_{-M}=\infty\}}.
\end{aligned}$$

By the monotone convergence theorem, the right hand side of eq. (B.4) satisfies:

$$\begin{aligned}
\limsup_{t\to\infty} \text{RHS of eq. (B.4)} &= \limsup_{t\to\infty} \mathbb{E}\left[\int_0^{t\wedge T\wedge T_{-M}} e^{-2pru} A\mathcal{X}_u\, m(du)\right] \\
&= \mathbb{E}\left[\limsup_{t\to\infty} \int_0^{t\wedge T\wedge T_{-M}} e^{-2pru} A\mathcal{X}_u\, m(du)\right] \\
&\geq \mathbb{E}\left[\limsup_{t\to\infty} \int_0^{t} e^{-2pru} A\mathcal{X}_u(\omega)\mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\, m(du)\right] \\
&\geq \mathbb{E}\left[\infty \cdot \mathbf{1}_{\{T\wedge T_{-M}=\infty\}}\right] \\
&= \infty \times \epsilon = \infty.
\end{aligned}$$

This contradicts the fact that the lim sup of LHS of eq. (B.4) is bounded.

□

We have now proved all three statements of Theorem 3.1.

Appendix C Speed of Extinction

In this section we prove Theorem 4.1, part 2. We use the same notations as in the proof of Theorem 4.1, part 1.

Proof. Applying Theorem A.9 to the martingale $M_n(\theta) = e^{\theta \bar{S}_n - n\psi(\theta)}$, we have

$$\mathbb{E}[e^{\theta \bar{S}_{t\wedge\tau} - (t\wedge\tau)\psi(\theta)}] = e^{\theta S_0}.$$

Recall that for every sample path ω, $S_t(\omega) \geq \bar{S}_t(\omega)$ and $\theta < 0$, then

$$\begin{aligned}
& \mathbb{E}\left[e^{\theta S_{t\wedge\tau} - (t\wedge\tau)\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \\
\leq & \mathbb{E}\left[e^{\theta \bar{S}_{t\wedge\tau} - (t\wedge\tau)\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \\
\leq & \mathbb{E}\left[e^{\theta \bar{S}_{t\wedge\tau} - (t\wedge\tau)\psi(\theta)}\right] \\
= & e^{\theta S_0}.
\end{aligned}$$

By Fatou's lemma we have:

$$\begin{aligned}
& \mathbb{E}\left[\liminf_{t\to\infty} e^{\theta S_{t\wedge\tau}-(t\wedge\tau)\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \\
\leq\ & \liminf_{t\to\infty} \mathbb{E}\left[e^{\theta S_{t\wedge\tau}-(t\wedge\tau)\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \\
\leq\ & e^{\theta S_0}.
\end{aligned}$$

This gives us a very useful inequality

$$\mathbb{E}\left[e^{\theta S_\tau-\tau\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \leq e^{\theta S_0},$$

which implies

$$\mathbb{E}\left[e^{\theta S_\tau-\tau\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\mathbf{1}_{\{\tau>t\}}\right] \leq \mathbb{E}\left[e^{\theta S_\tau-\tau\psi(\theta)}\mathbf{1}_{\{\tau<\infty\}}\right] \leq e^{\theta S_0}.$$

Recall that $S_\tau \leq 0$, $\theta < 0$ and $\psi(\theta) < 0$, then

$$\begin{aligned}
e^{\theta S_\tau}\mathbf{1}_{\{\tau<\infty\}} &\geq \mathbf{1}_{\{\tau<\infty\}}, \\
e^{-\tau\psi(\theta)}\mathbf{1}_{\{\tau>t\}} &\geq e^{-t\psi(\theta)}\mathbf{1}_{\{\tau>t\}}.
\end{aligned}$$

Therefore

$$\mathbb{P}[\tau<\infty, \tau>t]e^{-t\psi(\theta)} \leq e^{\theta S_0}.$$

Thus we conclude that

$$\mathbb{P}[\tau > t|\tau<\infty] \leq \frac{1}{\mathbb{P}[\tau<\infty]}e^{\theta S_0+t\psi(\theta)}.$$

Finally, we remark that one can find the best upper bound by minimizing the RHS over θ. □

Appendix D Endogenous Parameters

A natural way of endogenizing the parameters is to define them as functions of the current state of the world. In this section, instead of having p and r as fixed constants, we study the case when $p_{t+1} = H(\frac{Y_t}{X_t+Y_t+Z_t})$ and $r_{t+1} = Q(\frac{Y_t}{X_t+Y_t+Z_t})$, where H and Q are two continuous functions. In other words, the extent and intensity of repugnance may depend on the current proportion of drug despisers in the system. Unfortunately those techniques we employed previously do not carry over directly, therefore we present a heuristic analysis (and partial proofs) together with some simulations to provide a robustness check for our results.

It is clear that if the system eventually stabilizes, i.e. $\lim_{t\to\infty} p_t = p^*$, and $\lim_{t\to\infty} r_t = r^*$, then it has to be the case that $\lim_{t\to\infty}\frac{Y_t}{X_t+Y_t+Z_t} = p^*$, by the law of large numbers. Hence p^* satisfies the equation $p^* = H(p^*)$, i.e. p^* is a fixed point of H. Once p^* is determined, then $r^* = Q(p^*)$ is also determined. Therefore essentially, even though now p and r are endogenous, we can treat them as constants p^* and r^* in the long run. And Theorem 3.1 should still hold in this extension, if we replace p and r by p^* and r^*.

However, there is a complication: not all fixed points of H become the limit of p_t; there are also saddle points. To see this, define $v_t = \frac{Y_t}{X_t+Y_t+Z_t}$ and $N = X_t+Y_t+Z_t$.

$$\begin{aligned}
\mathbb{E}[v_{t+1}|X_t,Y_t,Z_t] - v_t &= H(v_t)\frac{Y_t+1}{N+1} + (1-H(v_t))\frac{Y_t}{N+1} - \frac{Y_t}{N} \\
&= \frac{Y_t+H(v_t)}{N+1} - \frac{Y_t}{N} \\
&= \frac{1}{N+1}(H(v_t)-v_t).
\end{aligned}$$

Therefore $\mathbb{E}[v_{t+1}|X_t, Y_t, Z_t] > v_t \Leftrightarrow H(v_t) > v_t$. If we want v_t to stabilize around some p^*, it has to be the case that when $v_t > p^*$, $\mathbb{E}[v_{t+1}|X_t, Y_t, Z_t] < v_t$ and when $v_t < p^*$, $\mathbb{E}[v_{t+1}|X_t, Y_t, Z_t] > v_t$, so v_t is pushed towards p^* in expectation. That means, for any v in a small neighborhood of p^*, we have $v < p^* \Leftrightarrow H(v) > v \quad (\star)$. If condition $(\star)$ is not satisfied, then v_t is pushed away from p^*, and p^* will be a saddle point, instead of a stabilizing limit of p_t.

In fact, the convergence of v_t is studied in Hill, Lane, and Sudderth [30] (see also Pemantle [35]). And their conclusions (with rigorous proofs) agree with our heuristic analysis. Specifically, Theorem 2.1 in their paper says that v_t converges almost surely (as a random variable). Then Corollary 3.1 in their paper implies that almost surely the limit of v_t satisfies $x = H(x)$. Finally Theorem 5.1 in their paper confirms that only the fixed points that satisfy condition $(\star)$ are in the support of the limit. Therefore we do have the convergence result for $\frac{Y_t}{X_t+Y_t+Z_t}$, $p_{t+1} = H(\frac{Y_t}{X_t+Y_t+Z_t})$ and $r_{t+1} = Q(\frac{Y_t}{X_t+Y_t+Z_t})$ (by the continuous mapping theorem). Unfortunately we can not rigorously prove the convergence of R_t. Although through simulations we do see that R_t converges to $\frac{1-p^*-p^*r^*}{p^*}$.

To summarize, if p^* is a fixed point of H and $(\star)$ is satisfied, then p^* is a potential stabilizing point of p_t. Given a market, if some of these stabilizing p^*'s satisfy $p^*(1 + r^* + Kr^*) < 1$, then the market will survive with a positive probability, and $\bar{R}^* = \frac{1-p^*-p^*r^*}{p^*}$ is one potential limit for R_t. (If multiple $\bar{R}^*$'s exist, then the limit of R_t depends on the realization.) Otherwise, if $p^*(1 + r^* + Kr^*) > 1$ for all such p^*, then the market becomes extinct almost surely. Below we provide some simulations to illustrate this result.

First, consider a linear example with $H(u) = 0.2 + 0.6u$, and $Q(u) = 0.1 + u^2$. It is clear that the unique fixed point of H is $p^* = 0.5$, and $(\star)$ is satisfied. And we can compute that $r^* = 0.35$ and $\bar{R}^* = 0.65$. When $K = 1$, the market is uncontrollable, Figure 2 below shows the convergence of p_t and r_t, and Figure 3 shows the convergence of $R_t = X_t/Y_t$. Only one out of forty paths reaches extinction, which agrees with the exponential decay pattern in Theorem 4.1. The threshold K (to be the borderline case) in this example is 13/7, we did 1000 simulations for $K = 2$ and all realizations die before time 10^8.

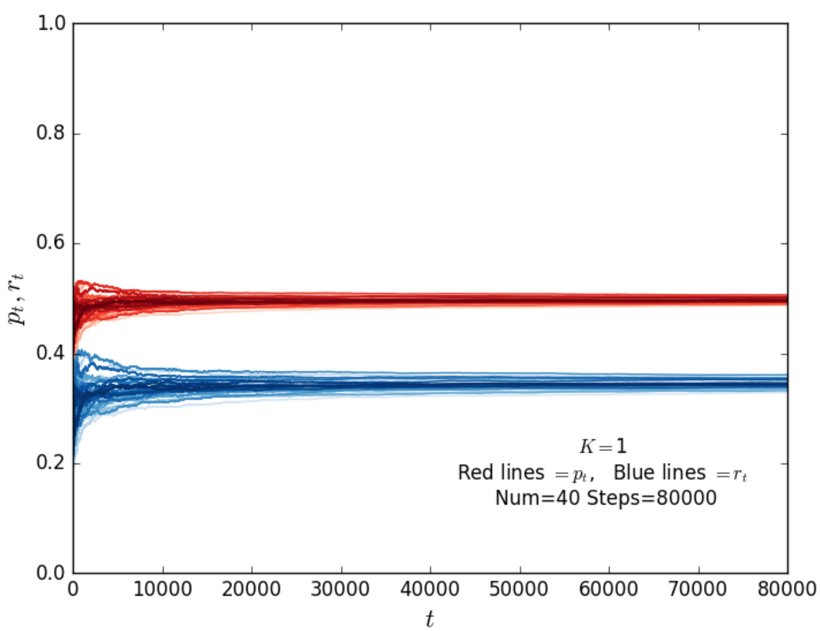

Figure 2: Convergence of p_t and r_t with 1 fixed point

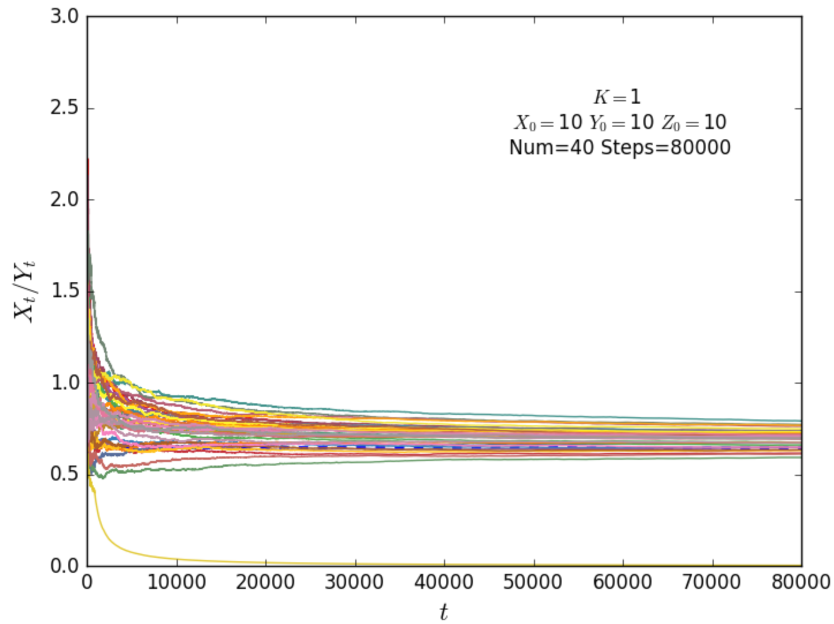

Figure 3: Convergence of R_t with 1 fixed point

To demonstrate condition ($\star$), consider $H(u) = 0.5 + 0.3sin(10u)$, $r = 0.1$, $K = 0.5$. There are three solutions of $H(u) = u$, they are (approximately) 0.362, 0.702, 0.798. From Figure 4 below, we can see that 0.362 and 0.798 satisfy ($\star$), while 0.702 does not. And indeed, Figure 6 shows that, the $\bar{R}^*$'s corresponding to 0.362 and 0.798, 1.66 and 0.153, are limits of realized R_t's, while $\bar{R}^* = 0.325$, corresponding to 0.702, is not.

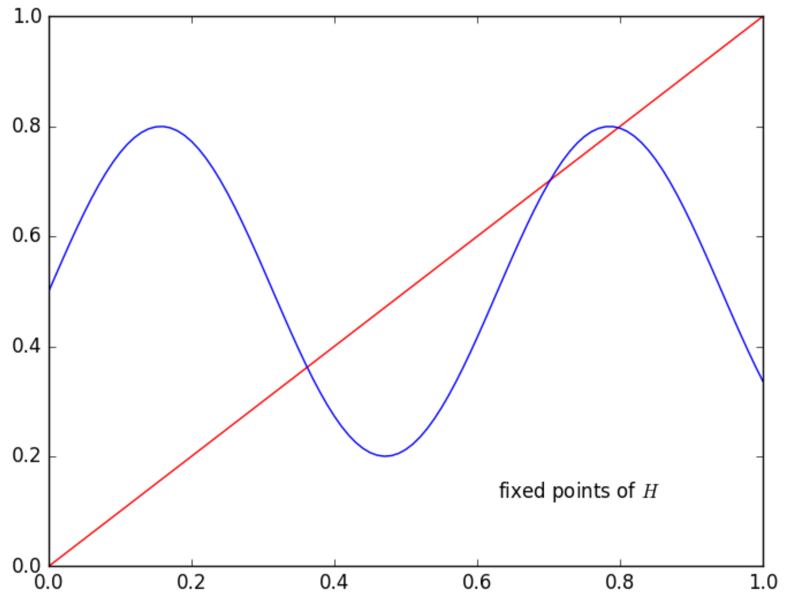

Figure 4: Fixed points

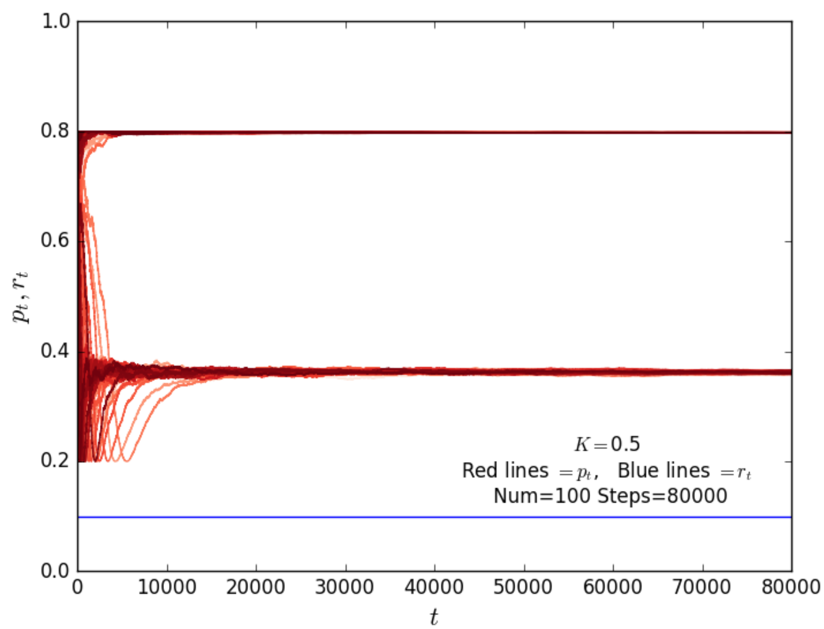

Figure 5: Convergence of p_t and r_t with 3 fixed points

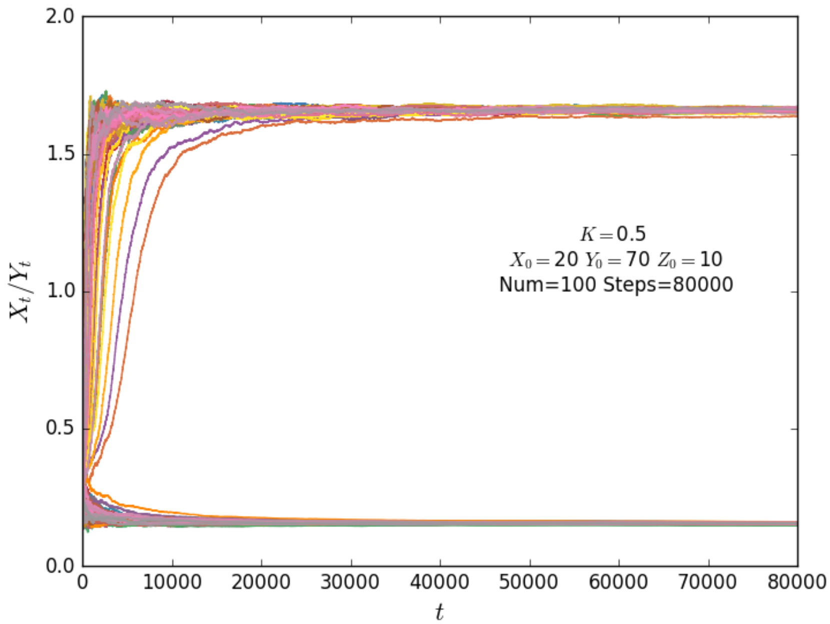

Figure 6: Convergence of R_t with 3 fixed points

Appendix E Extensions

In this part of appendix we discuss a few extensions to the base model.

E.1 Internal Dynamics

In the baseline model, only the newcomer influences the composition of the population, i.e. agents never change their type once they have entered the system. In this subsection, we describe how to incorporate internal dynamics, e.g. agents might change their type from Z to X, into our model.

Note that our embedding continuous process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)$ can be seen as a general population model, with reproduction rate:

$$A = \begin{pmatrix} A_{\mathcal{X},\mathcal{X}} & A_{\mathcal{X},\mathcal{Y}} & A_{\mathcal{X},\mathcal{Z}} \\ A_{\mathcal{Y},\mathcal{X}} & A_{\mathcal{Y},\mathcal{Y}} & A_{\mathcal{Y},\mathcal{Z}} \\ A_{\mathcal{Z},\mathcal{X}} & A_{\mathcal{Z},\mathcal{Y}} & A_{\mathcal{Z},\mathcal{Z}} \end{pmatrix}, \tag{E.1}$$

where each element of A represents the rate of reproduction between two types. For example, $A_{\mathcal{X},\mathcal{Y}}$ represents the reproduction rate of population $\mathcal{Y}$ from $\mathcal{X}$. In particular, the transition matrix of the base model in Section B is the following:

$$A = \begin{pmatrix} 1-p & p & 0 \\ 0 & pr & (1-p)r \\ 0 & 0 & 0 \end{pmatrix}. \tag{E.2}$$

Then internal dynamics can be modeled through modifying this A matrix: e.g. if drug users are actively being arrested, then $A_{\mathcal{X},\mathcal{X}}$ should be decreased; if drug neutrals can turn into drug users, then $A_{\mathcal{Z},\mathcal{X}} > 0$. The Perron–Frobenius theorem is a powerful and universal method for analyzing such a model: there exists a normalized positive left eigenvector μ associated with the maximum eigenvalue λ of A i.e.

$$\mu A = \lambda \mu,$$

and the population size at time t is about

$$(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t) \simeq \mu e^{At} = e^{\lambda t} \mu.$$

Therefore μ is the limiting proportion of the three types in the long run. See Athreya [4] for details.

E.2 Drug Dealers

One could also explicitly model drug dealers, in a similar fashion to the model in the previous subsection. More specifically, we may split $\mathcal{X}$ into $\mathcal{X}_1$ and $\mathcal{X}_2$, representing drug dealers and users respectively. Then the population dynamics may be characterized by a 4×4 matrix:

$$A = \begin{pmatrix} A_{\mathcal{X}_1,\mathcal{X}_1} & A_{\mathcal{X}_1,\mathcal{X}_2} & A_{\mathcal{X}_1,\mathcal{Y}} & A_{\mathcal{X}_1,\mathcal{Z}} \\ A_{\mathcal{X}_2,\mathcal{X}_1} & A_{\mathcal{X}_2,\mathcal{X}_2} & A_{\mathcal{X}_2,\mathcal{Y}} & A_{\mathcal{X}_2,\mathcal{Z}} \\ A_{\mathcal{Y},\mathcal{X}_1} & A_{\mathcal{Y},\mathcal{X}_2} & A_{\mathcal{Y},\mathcal{Y}} & A_{\mathcal{Y},\mathcal{Z}} \\ A_{\mathcal{Z},\mathcal{X}_1} & A_{\mathcal{Z},\mathcal{X}_2} & A_{\mathcal{Z},\mathcal{Y}} & A_{\mathcal{Z},\mathcal{Z}} \end{pmatrix}. \tag{E.3}$$

The supply and demand relation between drug dealers and users can be modeled through $A_{\mathcal{X}_1,\mathcal{X}_2}$ and $A_{\mathcal{X}_2,\mathcal{X}_1}$, and the competition effect between dealers can be characterized by $A_{\mathcal{X}_1,\mathcal{X}_1}$. The long run population composition is still determined by the Perron–Frobenius eigenvector.

E.3 Imperfect Information, Irrationality and General Decision Functions

In this paper, we have analyzed two processes in detail:

- The process $(X_t, Y_t, Z_t)_{t\geq 0}$ in discrete time, with the decision criterion for the potential drug user being $\frac{X_t}{Y_t} \geq Kr$. This implicitly assumes that potential drug users have perfect information about the ratio $\frac{X_t}{Y_t}$, and that they are fully rational.
- The embedding process $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}$ in continuous time as an auxiliary process. The dynamics of this process implicitly assumes that potential drug users always look for drugs. This kind of behavior may be caused by lack of information, or irrationality.

In practice, the ratio $\frac{X_t}{Y_t}$ may not be accessible to the public or may be distorted, and the newcomer may behave in certain irrational way. In general we can model the potential drug user's decision as:

$$\left(\frac{X_t}{Y_t}, K, r, \epsilon_{t+1}\right) \mapsto f\left(\frac{X_t}{Y_t}, K, r, \epsilon_{t+1}\right) \in \{0,1\}, \tag{E.4}$$

where $f = 1$ represents looking for drugs and $f = 0$ represents not looking for drugs. The quantity ϵ_{t+1} is a random variable at time $(t+1)$. The two processes above can be seen as two extreme cases: The process $(X_t, Y_t, Z_t)_{t\geq 0}$ is the case in which the potential drug user has perfect information of $\frac{X_t}{Y_t}$ and is rational

$$f_1\left(\frac{X_t}{Y_t}, K, r, \epsilon_{t+1}\right) = \mathbf{1}_{\left\{\frac{X_t}{Y_t} > Kr\right\}},$$

while the potential drug user in $(\mathcal{X}_t, \mathcal{Y}_t, \mathcal{Z}_t)_{t\geq 0}$ has

$$f_2\left(\frac{X_t}{Y_t}, K, r, \epsilon_{t+1}\right) = 1.$$

For another process $(\widetilde{X}_t, \widetilde{Y}_t, \widetilde{Z}_t)_{t\geq 0}$ with decision function f, if it is true that

$$f_1\left(\frac{x}{y}, K, r, \epsilon\right) \leq f\left(\frac{x}{y}, K, r, \epsilon\right) \leq f_2\left(\frac{x}{y}, K, r, \epsilon\right), \tag{E.5}$$

then we necessarily have (by the coupling argument, in a path by path sense)

$$\lim_{t\to\infty} \frac{X_t}{Y_t} \leq \lim_{t\to\infty} \frac{\widetilde{X}_t}{\widetilde{Y}_t} \leq \lim_{t\to\infty} \frac{\mathcal{X}_t}{\mathcal{Y}_t}. \tag{E.6}$$

Here is one specific example that might arise in the case of imperfect information/irrationality. Suppose due to lack of information, there is always a proportion θ of potential drug users willing to search for drugs regardless of $\frac{\widetilde{X}_t}{\widetilde{Y}_t}$, while others behave like rational agents with perfect information. That is, if we denote by $(\epsilon_t)_{t\geq 1}$ i.i.d. random variables independent of the process, and $\epsilon_t \sim \text{Uniform}[0, 1]$, then

$$f\left(\frac{\widetilde{X}_t}{\widetilde{Y}_t}, K, r, \epsilon_{t+1}\right) = \mathbf{1}_{\{\frac{\widetilde{X}_t}{\widetilde{Y}_t} > Kr\}} + \mathbf{1}_{\{\frac{\widetilde{X}_t}{\widetilde{Y}_t} \leq Kr\}}\mathbf{1}_{\{\epsilon_{t+1}\leq\theta\}}.$$

If we write down the transition probability, it is

$$\begin{aligned}
P\left((x+1, y, z)|(x, y, z)\right) &= (1-p)\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} + (1-p)\theta\frac{x}{x+ry}\mathbf{1}_{\{\frac{x}{y}\leq Kr\}}, \\
P\left((x, y+1, z)|(x, y, z)\right) &= p, \\
P\left((x, y, z+1)|(x, y, z)\right) &= (1-p)\frac{ry}{x+ry}\mathbf{1}_{\{\frac{x}{y}>Kr\}} + (1-p)(1-\theta\frac{x}{x+ry})\mathbf{1}_{\{\frac{x}{y}\leq Kr\}}.
\end{aligned}$$

We can do a similar asymptotic analysis as in Section 3, and see that this example also can be divided into different cases and the parameter θ plays an important role: (the complication here is simply that, if the ratio $\frac{\widetilde{X}_t}{\widetilde{Y}_t}$ can not be sustained above Kr, we may hit a steady state that θ proportion of uninformed/irrational agents always go for drugs, while others directly become neutral.)

1. Controllable case: $p > \max\left\{\frac{1}{1+r+Kr}, \frac{1}{1+r/\theta}\right\}$, then the limit is

$$\lim_{t\to\infty} \frac{\widetilde{X}_t}{\widetilde{Y}_t} = 0.$$

2. Weakly controllable case: $\frac{1}{1+r+Kr} < p < \frac{1}{1+r/\theta}$, then the black market is controlled (none of the (1-θ)-agents will try to find drugs) but will not become extinct, and the limit is

$$\lim_{t\to\infty} \frac{\widetilde{X}_t}{\widetilde{Y}_t} = \frac{(1-p)\theta - pr}{p}.$$

   (This case implicitly assumes $\theta(K+1) > 1$.)

3. Uncontrollable case: $p < \frac{1}{1+r+Kr}$, then there is a positive probability that the market survives just as in the base model, and

$$\lim_{t\to\infty} \frac{\widetilde{X}_t}{\widetilde{Y}_t} = \frac{(1-p)-pr}{p}.$$

However, depending on the situation, it may also converge to another limit:

(a) If $\theta(K+1) > 1$, the process always survives and another possible limit is (just as the weakly controllable case)

$$\lim_{t\to\infty} \frac{\widetilde{X}_t}{\widetilde{Y}_t} = \frac{(1-p)\theta-pr}{p}.$$

(b) If $\theta(K+1) < 1$, then if $\frac{1}{1+r/\theta} < p < \frac{1}{1+r+Kr}$, the process has a positive probability to become extinct. Otherwise, if $p < \frac{1}{1+r/\theta}$, another possible limit is again $\frac{(1-p)\theta-pr}{p}$.

E.4 Search Process

In the base model we assume that the newcomer would draw a person from the population uniformly random, and then ask for drugs. A possible generalization of this process is that he may have a different probability of meeting people of different types. For example, the probability that a newcomer meets a drug user, a drug despiser and a drug neutral may be $\frac{\alpha X_t}{\alpha X_t+\beta Y_t+\gamma Z_t}$, $\frac{\beta Y_t}{\alpha X_t+\beta Y_t+\gamma Z_t}$, and $\frac{\gamma Z_t}{\alpha X_t+\beta Y_t+\gamma Z_t}$ respectively (in the base model $\alpha = \beta = \gamma$). Then in the one-step-exploration analysis, the probability q to find drugs and become a drug user is

$$q = \frac{\alpha X_t}{\alpha X_t+\beta Y_t+\gamma Z_t}\cdot 1 + \frac{\beta Y_t}{\alpha X_t+\beta Y_t+\gamma Z_t}\cdot r\cdot 0 + \frac{\beta Y_t}{\alpha X_t+\beta Y_t+\gamma Z_t}\cdot(1-r)\cdot q + \frac{\gamma Z_t}{\alpha X_t+\beta Y_t+\gamma Z_t}\cdot q.$$

This equation gives us

$$q = \frac{\alpha X_t}{\alpha X_t + r\beta Y_t},$$

and the transition probability

$$\begin{aligned}
P\left((x+1,y,z)|(x,y,z)\right) &= (1-p)\frac{\alpha x}{\alpha x + r\beta y}\mathbf{1}_{\{\frac{x}{y}>Kr\frac{\beta}{\alpha}\}},\\
P\left((x,y+1,z)|(x,y,z)\right) &= p,\\
P\left((x,y,z+1)|(x,y,z)\right) &= (1-p)\frac{r\beta y}{\alpha x + r\beta y}\mathbf{1}_{\{\frac{x}{y}>Kr\frac{\beta}{\alpha}\}} + (1-p)\mathbf{1}_{\{\frac{x}{y}\le Kr\frac{\beta}{\alpha}\}}.
\end{aligned}$$

The asymptotic limit between drug users and drug despisers when the market survives is then

$$\bar{R} = \frac{(1-p)\alpha - pr\beta}{p\alpha}.$$

The analysis of this generalization is completely similar to the base model.

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